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Original Article

Dynamic similarity criteria for simple cases of building and structure aerodynamics

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Abstract: This work concerns the dynamic similarity criteria of various phenomena occurring in the aerodynamics of buildings and structures, originally derived from the ratios of forces and force moments affecting these phenomena. This paper is a continuation of [12], which addresses the foundations of dynamic similarity criteria formulated in this manner. At the end of [12], an authorial method and procedure for determining dynamic similarity criteria in fluid-solid interaction issues are presented. This method serves as the basis for the formulations and considerations of dynamic similarity criteria discussed further for various practical problems encountered in simple cases of building and structure aerodynamics, including self-exciting vibrations and wind-induced vibrations.

Keywords: dynamic similarity criteria, wind-induced vibrations, aerodynamics, aeroelasticity

1. Introduction

The paper concerns the determination and analysis of dynamic similarity criteria for simple cases of wind-induced vibrations encountered in the aerodynamics of buildings and structures, including self-exciting vibrations and vibrations caused by turbulent wind.

Dynamic similarity criteria were originally derived from the ratios of forces and force moments affecting the phenomena considered. This paper is a continuation of [12], which addresses the foundations of dynamic similarity criteria formulated in this way. At the end of [12], an authorial method and procedure for determining dynamic similarity criteria in fluid-solid interaction issues are presented. This method forms the basis for the formulations and considerations of dynamic similarity criteria discussed further for various practical problems encountered in simple cases of building and structure aerodynamics.

The considerations related to these specific cases of building and structure aerodynamics are preceded by a relevant literature review.

2. Across-wind galloping

The large-amplitude across-wind oscillation of iced power line conductors provides a classic example of galloping. It can also be a potential issue for tall, flexible prismatic towers and flexible cylinders or prisms with certain types of cross-sections (e.g., rectangular sections, D-sections). The amplitudes of aeroelastic galloping oscillations can reach 1 to 10 or more times the cross-sectional dimensions of the body.

The flow speeds required for galloping motions are typically much higher than those for vortex lock-in oscillations. Flow reattachment, which occurs in vortex lock-in and flutter phenomena, does not occur in the case of galloping. Instead, completely separated flows characterize galloping motion, resulting in the absence of vortex-induced effects on the body.

The foundations of galloping theory appeared early in [19] and [21]. Novak and Tanaka [20] investigated the effect of turbulence on galloping instability, while Nakamura and Tomonari [18] provided a detailed study on the galloping of rectangular prisms in turbulent flow.

Galloping is primarily governed by quasi-steady forces, which depend on the relative angle of wind attack to the structural cross-section, moving across the wind with velocity $\dot{y}$.

Let us consider a section of a prismatic body in a smooth oncoming airflow with velocity V (Fig. 1)

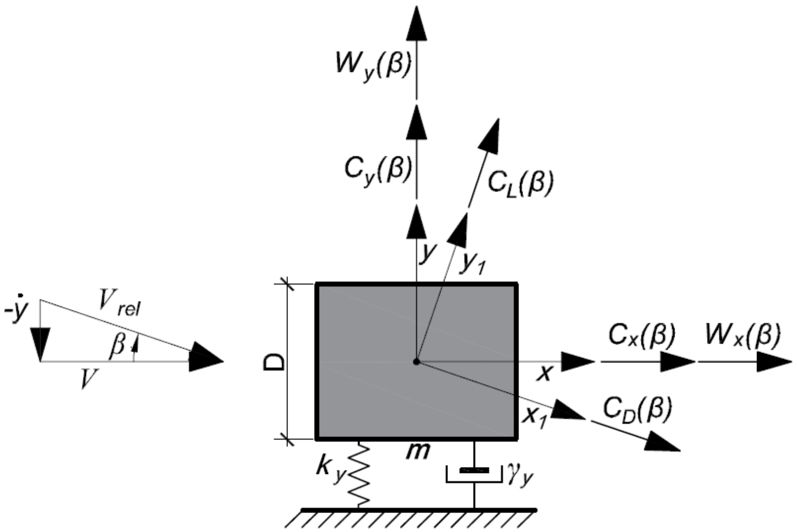

Fig. 1. One-degree-of-freedom model of across-wind galloping (*source*: authors)

The magnitude of the relative velocity of the flow with respect to the moving body is:

$$V_{rel} = \sqrt{(V^2 + \dot{y}^2)} \tag{1}$$

The angle of wind attack β is then:

$$\beta = tan^{-1}(-\dot{y}/V) \tag{2}$$

The across-wind force W_y in the analysed case is given by [5,8]:

$$W_y(\beta) = \frac{1}{2}\rho V^2 D C_{Wy}(\beta); \qquad C_{Wy}(\beta) = \frac{V_{rel}^2}{V^2}[C_L(\beta)cos\beta + C_D(\beta)sin\beta] \tag{3}$$

where ρ is the air mass density.

If the body is treated as a one-degree-of-freedom system, its equation of motion can be written in the usual form:

$$m\ddot{y} + 2m\gamma_y\omega_y\dot{y} + k_y y = W_y(\beta) = W_y\left(\frac{\dot{y}}{V}\right) \tag{4}$$

where: m, γ_y, k_y and $\omega_y^2 = \frac{k_y}{m} = 4\pi^2 f_y^2$ are the mechanical parameters of the system. For the linear case, the force $W_y\left(\frac{\dot{y}}{V}\right)$ can be expressed as [5,8]:

$$W_y\left(\frac{\dot{y}}{V}\right) \cong \frac{1}{2}\rho V^2 D\left[C_L|_{\beta=0} + \left(\frac{\partial C_L}{\partial\beta} + C_D\right)\Big|_{\beta=0}\left(-\frac{\dot{y}}{V}\right)\right] \tag{5}$$

where C_L and C_D are the steady lift and drag aerodynamic coefficients, respectively. Then, eq. (4) can be rewritten as:

$$m\ddot{y} + 2m\omega_y\left[\gamma_y + \frac{\rho V D}{4m\omega_y}\left(\frac{\partial C_L}{\partial\beta} + C_D\right)\Big|_{\beta=0}\right]\dot{y} + k_y y = \frac{1}{2}\rho V^2 D C_L\Big|_{\beta=0} \tag{6}$$

Assuming dimensional base of (ρ, V, D), it can be expressed as:

$$\check{t} = \frac{V}{D}t; \qquad \check{y}(\check{t}) = \frac{y(\check{t}\frac{D}{V})}{D}; \qquad \frac{dy(t)}{dt} = V\frac{d\check{y}(\check{t})}{d\check{t}}; \qquad \frac{d^2y(t)}{dt^2} = \frac{V^2}{D}\frac{d^2\check{y}(\check{t})}{d\check{t}^2} \tag{7}$$

Let us assume, moreover, the following similarity numbers:

$$Sr_y = \frac{f_y D}{V} \text{ – called kinematic Strouhal number} \tag{8}$$

$$M\rho_y = \frac{\rho D^2}{2m} \text{ – called dimensionless parameter of mass} \tag{9}$$

Then eq. (6) can be rewritten in dimensionless form as:

$$\frac{d^2\check{y}}{d\check{t}^2} + 4\pi Sr_y\left[\gamma_y + \frac{1}{4\pi}\frac{M\rho_y}{Sr_y}\left(\frac{\partial C_L}{\partial\beta} + C_D\right)\Big|_{\beta=0}\right]\frac{d\check{y}}{d\check{t}} + 4\pi^2 Sr_y^2\check{y} = M\rho_y C_L\Big|_{\beta=0} \tag{10}$$

The particular dimensionless factors occurring in the dimensionless eq. (10) can be interpreted as follows:

$$4\pi Sr_y\left[\gamma_y + \frac{1}{4\pi}\frac{M\rho_y}{Sr_y}\left(\frac{\partial C_L}{\partial\beta} + C_D\right)\Big|_{\beta=0}\right] \tag{11}$$

a measure of the ratio of the total damping force (i.e., structural and aerodynamic damping force) to the inertial force:

$4\pi^2 Sr_y^2$ – a measure of the ratio of the elastic force to the inertial force (12)

$M\rho_y C_L\big|_{\beta=0}$ – a measure of the ratio of the aerodynamic force to the inertial force (13)

The quasi-steady aerodynamic coefficients $C_D(\beta)$ and $C_L(\beta)$ are typically determined in quasi-steady tests performed in wind tunnels as functions of the dimensionless parameters characterizing the oncoming airflow i.e., input parameters $(\widetilde{IN})$ and the geometrical parameters of the body/object $(\breve{G})$.

From eq. (10), it follows that the necessary condition for the onset of across-wind galloping instability is the well-known Glauert–Den Hartog criterion in the form:

$$\left(\frac{\partial C_L}{\partial \beta} + C_D\right)\bigg|_{\beta=0} < 0 \tag{14}$$

and that the critical velocity V_c^g for the onset of across-wind galloping instability is given by [2,8]:

$$Vr_c^g = \frac{1}{Sr_c^g} = \frac{V_c^g}{f_y D} = -\frac{4\pi\gamma_y}{M\varrho_y}\left(\frac{\partial C_L}{\partial \beta} + C_D\right)\bigg|_{\beta=0} \tag{15}$$

where f_y is the natural frequency of the system.

3. Torsional divergence and torsional galloping

In across-wind galloping, changes in the angle of wind attack induced by vibrations are a function of the across-wind displacement velocity. In torsional divergence and torsional galloping, the angle of wind attack changes with the angular position ε and also with the angular velocity $\dot{\varepsilon} = \frac{d\varepsilon}{dt}$ of the section of the body, treated as a one-degree-of-freedom system with respect to torsion (torque). Thus:

$$\beta = \varepsilon - f(\dot{\varepsilon}) \cong \varepsilon - \frac{R\dot{\varepsilon}}{V} \tag{16}$$

where R is the characteristic radius for the given cross-section and pivot position.

According to the quasi-steady theory, the torsional moment W_ε acting on the section about the pivot is given by:

$$W_\varepsilon(\beta) = \frac{1}{2}\rho V^2 D^2 C_{W\varepsilon}(\beta); \qquad C_{W\varepsilon}(\beta) = C_{W\varepsilon}(\varepsilon, \dot{\varepsilon}, R) \tag{17}$$

In the analyzed case, the equation of motion for the torsional response of the section is (Fig. 2):

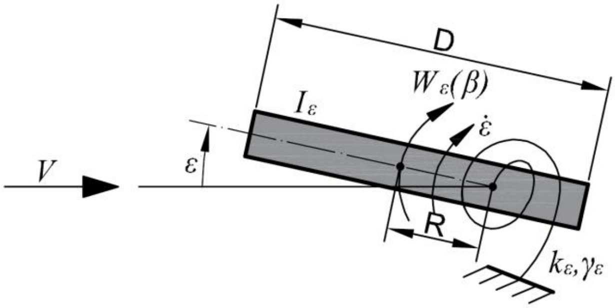

Fig. 2. One-degree-of-freedom model of torsional galloping (*source:* authors)

$$I_\varepsilon \ddot{\varepsilon} + 2\gamma_\varepsilon I_\varepsilon \gamma_\varepsilon \omega_\varepsilon \dot{\varepsilon} + k_\varepsilon \varepsilon = \frac{1}{2}\rho V^2 D^2 C_{W\varepsilon}(\varepsilon, \dot{\varepsilon}, R) \tag{18}$$

where: $I_\varepsilon, \gamma_\varepsilon, k_\varepsilon$ and $\omega_\varepsilon^2 = \frac{k_\varepsilon}{I_\varepsilon} = 4\pi^2 f_\varepsilon^2$ are the mechanical parameters of the system.

For small angles of wind attack, the coefficient $C_{W\varepsilon}$ on the right-hand side of this equation can be linearized for $\beta \ll 1$, as [2,8]:

$$C_{W\varepsilon}(\varepsilon, \dot{\varepsilon}, R) \cong C_M|_{\beta=0} + \left.\frac{\partial C_M}{\partial \beta}\right|_{\beta=0} \left(\varepsilon - \frac{R\dot{\varepsilon}}{V}\right) \tag{19}$$

where $C_M = C_{W\varepsilon}$ is the steady aerodynamic moment coefficient. Then eq. (18) takes the form:

$$I_\varepsilon \ddot{\varepsilon} + \left(2I_\varepsilon \gamma_\varepsilon \omega_\varepsilon + \frac{1}{2}\rho V R D^2 \left.\frac{\partial C_M}{\partial \beta}\right|_{\beta=0}\right)\dot{\varepsilon} + \left(k_\varepsilon - \frac{1}{2}\rho V^2 D^2 \left.\frac{\partial C_M}{\partial \beta}\right|_{\beta=0}\right)\varepsilon =$$

$$= \frac{1}{2}\rho V^2 D^2 C_M|_{\beta=0} \tag{20}$$

This equation exhibits two modes of instability. The first, called torsional divergence, when the sum of the structural and aerodynamic torsional stiffness terms becomes zero, leading to static-type instability. The second, called torsional galloping, when the coefficient of the $\dot{\varepsilon}$ term crosses zero. The necessary condition for the onset of this phenomenon is: $\left(R\partial C_M/\partial\beta|_{\beta=0}\right) < 0$. The critical velocities for torsional divergence V_c^{td} and torsional galloping V_c^{tg} are given, respectively, by:

$$V_c^{td} = \sqrt{\frac{2k_\varepsilon}{\rho D^2 \left.\frac{\partial C_M}{\partial \beta}\right|_{\beta=0}}} \tag{21}$$

$$V_c^{tg} = -\frac{4I_\varepsilon(2\pi f_\varepsilon)\gamma_\varepsilon}{\rho D^2 R \left.\frac{\partial C_M}{\partial \beta}\right|_{\beta=0}} \tag{22}$$

where f_ε is the natural frequency of the system. A more detailed analysis of these phenomena can be found in papers [5] and [17].

Proceeding similarly as before, the equation of motion can be brought into dimensionless form:

$$\frac{d^2\check{\varepsilon}(\check{t})}{d\check{t}^2} + \left(4\pi\gamma_\varepsilon Sr_\varepsilon + M\rho_\varepsilon \check{R}\left.\frac{\partial C_M}{\partial\beta}\right|_{\beta=0}\right)\frac{d\check{\varepsilon}(\check{t})}{d\check{t}} + \left(4\pi^2 Sr_\varepsilon^2 - M\rho_\varepsilon \left.\frac{\partial C_M}{\partial\beta}\right|_{\beta=0}\right)\check{\varepsilon}(\check{t}) =$$

$$= M\rho_\varepsilon C_M|_{\beta=0} \quad (23)$$

where:

$$Sr_\varepsilon = \frac{Df_\varepsilon}{V}; \qquad M\rho_\varepsilon = \frac{\rho D^4}{2I_\varepsilon}; \qquad \check{R} = \frac{R}{D} \quad (24)$$

The quasi-steady aerodynamic coefficient is generally a function of dimensionless parameters characterizing the input ($\widetilde{IN}$) and object ($\check{O}$) (e.g., turbulence intensity of the oncoming air I_v and ($\check{G}$) – set of geometrical object parameters).

The factors in the respective terms of eq. (23) represent measures of the ratios of the corresponding moments of forces.

4. Flutter

Flutter is an aeroelastic phenomenon that occurs in flexible bodies with relatively elongated cross-sectional shapes in plan [26,27]. This phenomenon involves oscillations with amplitudes that increase over time, potentially leading to catastrophic structural failure. Flutter, like other aeroelastic phenomena, requires solving equations of motion that involve, in particular: inertial forces, mechanical damping, elastic constraints, and aerodynamic forces (including self-excited forces). These parameters depend on the structure of the oncoming airflow as well as the shape and motion of the body. Flutter is distinct from vortex-induced lock-in oscillations. The latter involves aeroelastic flow-structure interactions that occur only at characteristic resonant velocities – i.e., those at which the vortex shedding frequency matches or is close to the structure's natural frequency. For velocities higher than those at which lock-in occurs, oscillations are much weaker than during lock-in itself. In contrast, it is observed that for velocities exceeding those at which flutter occurs, the oscillation amplitude increases monotonically with velocity.

The term "flutter" refers to a class of aeroelastic phenomena that can be further categorized using additional qualifying terms, such as classical flutter, single-degree-of-freedom flutter, panel flutter, etc. These terms were originally used in aerospace engineering, though some are also applied in wind engineering.

Classical flutter is a coupled motion in a system with at least two degrees of freedom, i.e., rotational (torsional) mode ε and across-wind displacement (bending) mode y, occurring at a single frequency different from the natural frequencies of the system. The coupling effects may arise from fluid force terms or from structural inertia or stiffness terms.

Single-degree-of-freedom flutter concerns oscillations in either torsion or across-wind motion and can be referred to as torsional or across-wind instability (galloping). This category includes phenomena such as stall flutter of airfoils (similar to across-wind galloping in a hard oscillator form), stop-sign flutter of traffic signs, torsional vibrations of suspended bridge spans, and others.

The term "panel flutter" refers to sustained oscillations of panels – such as the sides of large rockets – caused by the high-speed airflow along the panel's surface. While panel flutter typically does not occur in wind engineering, it could appear in cases involving large-span, lightweight suspended roofs. Related phenomena include flutter of taut canvas covers and flag flutter.

Let us consider the formulation of the two-dimensional bridge flutter problem in smooth flow as an example. The self-excited forces due to relatively small oscillations of the bridge deck can be characterized by fundamental functions known as flutter aerodynamic derivatives. As noted earlier, in the case of galloping, the self-excited forces are fully described by steady-state derivatives of aerodynamic coefficients of the type $\frac{dC}{d\beta}$, which can be obtained from measurements on a fixed body. In contrast, flutter derivatives depend on the oscillation frequency and must be determined from measurements on an oscillating body.

Bridge decks are typically symmetrical, meaning that their elastic and mass centers coincide. The dependence of flutter derivatives on the oscillation frequency f of the fluttering body can be expressed in terms of the non-dimensional reduced frequency.

$$K = \frac{2\pi B f}{V} \tag{25}$$

where B is the width of the deck and V is the mean wind flow velocity. If the horizontal displacement of the deck is also considered, the equations of motion for a two-dimensional section of a symmetrical bridge deck with linear viscous damping and elastic restoring forces in smooth flow can be expressed as [26,27]:

$$m\ddot{y} + c_y\dot{y} + k_y y = w_y \tag{26}$$

$$I\ddot{\varepsilon} + c_\varepsilon\dot{\varepsilon} + k_\varepsilon \varepsilon = w_\varepsilon \tag{27}$$

$$m\ddot{x} + c_x\dot{x} + k_x x = w_x \tag{28}$$

where y, ε and x are the vertical displacement, torsional angle, and horizontal displacement, respectively. A unit span is subjected to the aerodynamic lift w_y, moment w_ε and drag w_x and has mass m, moment of inertia I, vertical, torsional and horizontal restoring forces with stiffness k_y, k_ε, k_x respectively, along with viscous damping coefficients c_y, c_ε, c_x. The mathematical expressions for the aeroelastic actions are typically written as follows:

$$w_y = \frac{1}{2}\rho V^2 B \begin{bmatrix} KY_1^*(K)\frac{\dot{y}}{V} + KY_2^*(K)\frac{B\dot{\varepsilon}}{V} + K^2Y_3^*(K)\varepsilon + K^2Y_4^*(K)\frac{y}{B} + \\ +KY_5^*(K)\frac{\dot{x}}{V} + K^2Y_6^*(K)\frac{x}{B} \end{bmatrix} \tag{29}$$

$$w_\varepsilon = \frac{1}{2}\rho V^2 B^2 \begin{bmatrix} KE_1^*(K)\frac{\dot{y}}{V} + KE_2^*(K)\frac{B\dot{\varepsilon}}{V} + K^2E_3^*(K)\varepsilon + K^2E_4^*(K)\frac{y}{B} + \\ +KE_5^*(K)\frac{\dot{x}}{V} + K^2E_6^*(K)\frac{x}{B} \end{bmatrix} \tag{30}$$

$$w_x = \frac{1}{2}\rho V^2 B \begin{bmatrix} KX_1^*(K)\frac{\dot{x}}{V} + KX_2^*(K)\frac{B\dot{\varepsilon}}{V} + K^2X_3^*(K)\varepsilon + K^2X_4^*(K)\frac{x}{B} + \\ + KX_5^*(K)\frac{\dot{y}}{V} + K^2X_6^*(K)\frac{y}{B} \end{bmatrix} \tag{31}$$

Terms proportional to $\ddot{y}$, $\ddot{\varepsilon}$ and $\ddot{x}$ (i.e., added mass terms, reflecting the forces due to the body's motion that result in fluid accelerations around the body) do not appear in the equations above, as these terms are negligible in wind engineering applications. The terms involving y and x account for changes in the vibration frequency of the body due to aeroelastic effects, while the terms in ε reflect the influence of the angle of attack noted earlier. The quantities $\frac{\dot{y}}{V}$ and $\frac{B\dot{\varepsilon}}{V}$ correspond to the effective angle of wind attack and are non-dimensional. The coefficients Y_i^*, E_i^* and X_i^* are known as Scanlan flutter derivatives and are also non-dimensional. In the case of the classical flutter phenomenon, the mathematical model of the problem is simpler.

The basic relationships describing this case are as follows: consider a section of a slender structure treated as a two-degrees-of-freedom mechanically linear system, as shown in Fig. 3, subjected to the action of a smooth oncoming flow. The vertical displacement and the torsional angle are denoted by y and ε, respectively. The vertical aerodynamic force w_y and the torsional aerodynamic moment w_ε refer to the elastic axes y, ε which have their origin at the elastic centre (EC), also known as the shear centre or centre of torsion, i.e., the point about which a vertical static force produces displacement but no torsion.

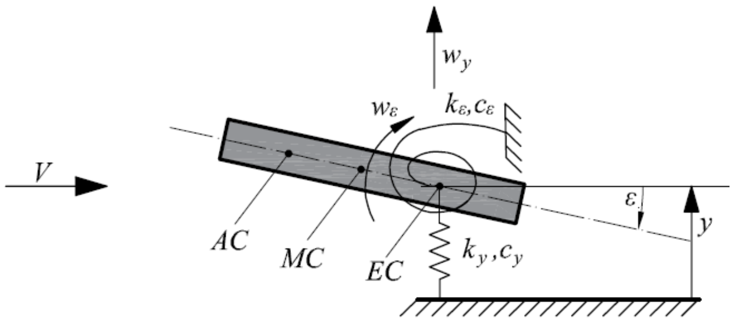

Fig. 3. Two-degrees-of-freedom model of classical flutter (*source:* authors)

A unit span of the system has a mass m, mass moment of inertia I, static unbalance S (equal to the product of mass m and the distance a separating the centre of mass MC from the elastic centre EC), along with vertical and torsional spring constants k_y and k_ε, respectively, and coefficients of vertical and torsional viscous damping c_y and c_ε, respectively. For such a system, the equations of motion can be expressed as [4]:

$$m\ddot{y} + S\ddot{\varepsilon} + c_y\dot{y} + k_y y = w_y \quad (32)$$

$$S\ddot{y} + I\ddot{\varepsilon} + c_\varepsilon\dot{\varepsilon} + k_\varepsilon\varepsilon = w_\varepsilon \quad (33)$$

For small oscillations, the self-excited aerodynamic lift w_y and moment w_ε on a bluff body may be treated as linear with respect to y and ε and their first two derivatives. Under this assumption, it is possible to measure the aerodynamic coefficients, for example, through special wind tunnel tests. Such experiments show that, similar to the case of airfoils, the aerodynamic coefficients of a bluff body are functions of the reduced velocity $V_r = V/(fB)$ of the oncoming flow.

Many different expressions for w_y and w_ε can be found in the literature (see, for example, [26,27]). The most commonly used form of these expressions is given by:

$$w_y = \frac{1}{2}\rho V^2 B\left[KY_1^*(K)\frac{\dot{y}}{V} + KY_2^*(K)\frac{B\dot{\varepsilon}}{V} + K^2Y_3^*(K)\varepsilon\right] \tag{34}$$

$$w_\varepsilon = \frac{1}{2}\rho V^2 B^2\left[KE_1^*(K)\frac{\dot{y}}{V} + KE_2^*(K)\frac{B\dot{\varepsilon}}{V} + K^2E_3^*(K)\varepsilon\right] \tag{35}$$

where the reduced frequency K is defined by eq. (25); B is the chord, deck width, or along-wind dimension of the section; V is a steady approaching wind speed; and f is the frequency of oscillations. The aerodynamic coefficients Y_i^* and E_i^* $(i = 1, 2, 3)$ are non-dimensional, nonlinear functions of K. References [24,25] discuss various experimental techniques for obtaining aerodynamic coefficients Y_i^* and E_i^*. Taking into account eqs. (34) and (35) and noting that:

$$t = \frac{B}{V}\check{t}; \quad dt = \frac{B}{V}d\check{t}; \quad dt^2 = \left(\frac{B}{V}\right)^2 d\check{t}^2 \tag{36}$$

$$\frac{d^2y(t)}{dt^2} = \frac{V^2}{B}\frac{d^2\check{y}(\check{t})}{d\check{t}^2}; \quad \frac{dy(t)}{dt} = V\frac{d\check{y}(\check{t})}{d\check{t}}; \quad y(t) = D\check{y}(\check{t}) \tag{37}$$

$$\frac{d^2\varepsilon(t)}{dt^2} = \frac{V^2}{B^2}\frac{d\check{\varepsilon}(\check{t})}{d\check{t}^2}; \quad \frac{d\varepsilon(t)}{dt} = \frac{V}{B}\frac{d\check{\varepsilon}(\check{t})}{d\check{t}}; \quad \varepsilon(t) = \check{\varepsilon}(\check{t}) \tag{38}$$

the equations of motion (32) and (33) can be rewritten in the following dimensionless form:

$$\frac{d^2\check{y}(\check{t})}{d\check{t}^2} + \frac{S}{mB}\frac{d^2\check{\varepsilon}(\check{t})}{d\check{t}^2} + \frac{c_yB}{mV}\frac{d\check{y}}{d\check{t}} + \frac{k_yB^2}{mV^2}\check{y}(\check{t}) =$$

$$= \frac{\rho B^2}{2m}\left[KY_1^*(K)\frac{d\check{y}(\check{t})}{d\check{t}} + KY_2^*(K)\frac{d\check{\varepsilon}(\check{t})}{d\check{t}} + K^2Y_3^*(K)\check{\varepsilon}(\check{t})\right] \tag{39}$$

$$\frac{SB}{I}\frac{d^2\check{y}(\check{t})}{d\check{t}^2} + \frac{d^2\check{\varepsilon}(\check{t})}{d\check{t}^2} + \frac{c_\varepsilon B}{IV}\frac{d\check{\varepsilon}(\check{t})}{d\check{t}} + \frac{k_\varepsilon B^2}{IV^2}\check{\varepsilon}(\check{t}) =$$

$$= \frac{\rho B^4}{2I}\left[KE_1^*(K)\frac{d\check{y}(\check{t})}{d\check{t}} + KE_2^*(K)\frac{d\check{\varepsilon}(\check{t})}{d\check{t}} + +K^2E_3^*(K)\check{\varepsilon}(\check{t})\right] \tag{40}$$

In eqs (39) and (40), the following dimensionless monomials or dimensionless functions appear:

$$\frac{S}{mB}, \frac{c_yB}{mV}, \frac{k_yB^2}{mV^2}, \frac{\rho B^2}{2m} \; KY_1^*(K), KY_2^*(K), K^2Y_3^*(K) \tag{41}$$

$$\frac{SB}{I}, \frac{c_\varepsilon B}{IV}, \frac{k_\varepsilon B^2}{IV^2}, \frac{\rho B^4}{2I}, KE_1^*(K), KE_2^*(K), K^2E_3^*(K) \tag{42}$$

These dimensionless quantities or functions, together with the dimensionless output quantities $\check{y}(\check{t})$, $\check{\varepsilon}(\check{t})$ and $\check{t}$, constitute the similarity criteria for the analysed problem.

Due to the dependence of the aerodynamic coefficients Y_i^* and E_i^* upon K, the solution to the flutter problem can be obtained using an iterative procedure. A typical solution method is as follows: for a chosen value of K, with corresponding values of Y_i^* and E_i^* taken from experimentally determined functions, the solution for y and ε is sought in the form of quantities proportional to $e^{j\omega t}$. As a result, the complex angular frequency is obtained in the form $\omega = \omega_1 + j\omega_2$. If $\omega_2 \neq 0$, a new value of K must be chosen, and the procedure is repeated until the imaginary part $\omega_2 \cong 0$, so that $\omega \cong \omega_1$. Let K_c be the value of K for which $\omega = \omega_1$. Then, the critical flutter velocity V_c^f is given by:

$$V_c^f = \frac{B\omega_1}{K_c} \tag{43}$$

The critical flutter velocity depends on the mass, damping, and stiffness parameters of the system, as well as on the geometrical relationships between the elastic centre EC, mass centre MC and aerodynamic centre AC. The aerodynamic centre is typically defined as the point of application of the resultant mean aerodynamic force and, for a given cross-section, is a function of the angle of wind attack.

There is also an alternative approach for modelling the aerodynamic lift w_y and moment w_ε: the quasi-steady approach. In this method, it is assumed that the expressions for w_y and w_ε are the same as in the steady case (i.e., without oscillations), provided that the steady angle of wind attack θ is replaced by the relative, instantaneous angle of wind attack θ_r. This relative angle is defined as $\theta_r = \theta_r(\varepsilon, \dot{\varepsilon}, \dot{y}) \cong \varepsilon + \dot{y}/V + R\dot{\varepsilon}/V$ where R is the characteristic radius for the given cross-section and the position of the EC. In this case, flutter derivatives are independent of the oscillation frequency f. Moreover, similar to the cases of galloping and torsional galloping, the formulae describing w_y and w_ε can be linearized (see, for example, papers [6-8] for motionless slender structures, moving slender structures, and real structures, respectively).

A steady oncoming wind, especially under strong wind conditions, is a very rare phenomenon. Typically, unsteady wind conditions are observed. In such cases, apart from self-excited wind action resulting from positive feedback between the airflow and structural motion, buffeting wind action also occurs. If the oscillations of the structure in each responding mode are relatively small compared to its characteristic transverse dimension, it can be assumed that the resulting wind action, caused by incident turbulence and wind-structure interaction, is a superposition of buffeting action and self-excited action. Naturally, the aerodynamic coefficients used in expressions describing the self-excited wind action must depend on the parameters of incident turbulence (e.g., turbulence intensity $I_v = \sigma_v/V$, where σ_v is the standard deviation of wind velocity fluctuations).

Most wind-structure interactions are highly complex and remain poorly understood. The mathematical models describing these phenomena are usually semi-empirical, incorporating just enough parameters to capture their most significant observed characteristics.

5. Critical vortex excitation of a slender cylindrical structure section in the case of lock-in phenomenon occurrence

The shedding of vortices in the wake of a body generates fluctuating lateral forces. As long as the motions are sufficiently small, they do not influence the vortex shedding.

When the Strouhal frequency of vortex shedding is:

$$f_v = \frac{St \cdot V}{D} \tag{44}$$

where: St – the Strouhal number, V – mean air onflow velocity, D – characteristic aerodynamic width, e.g. the diameter of a cylinder. When the Strouhal frequency of vortex shedding approaches the natural frequency f_y of the transverse oscillation of the cylinder, and the vibration amplitudes exceed a certain minimum level, an intensive interaction mechanism between the vibrations and vortex shedding is activated. This mechanism, in turn, controls the lateral forces on the cylinder. One of the primary effects of this feedback mechanism is the synchronization of the regular shedding frequency of the vortices with the oscillation frequency, known as the lock-in phenomenon. A corresponding critical velocity, close to the onset velocity of lock-in, is defined based on the resonance criterion as:

$$V_{vc} = \frac{f_y D}{St} \tag{45}$$

where f_y is the eigenfrequency of the system.

Experiments show that this condition occurs not only at the flow speed V_{vc} but also at any speed V within an interval $\frac{f_y D}{St} - \Delta V < V < \frac{f_y D}{St} + \Delta V$, where $\Delta V / V$ depends on the cross-sectional shape and mechanical damping and is typically on the order of a few percent. Within this interval, the vortex shedding frequency no longer follows the dependence in eq. (44) but instead aligns itself with the body's frequency f_y.

The aeroelastic effect occurs when the flow influences the motion of the body, and the body's motion, in turn, affects the flow, leading to synchronization of the vortex shedding frequency with the body's vibration frequency – commonly known as the lock-in effect.

Oscillations of the cylinder induced by vortex shedding can trigger a second aeroelastic effect: increased correlation of lateral forces along the span. In the case of an infinitely rigid cylinder, the vortex-induced lateral forces per unit span at different points along the cylinder are imperfectly correlated. However, when the cylinder oscillates due to lateral forces induced by the vortices, these oscillations enhance the correlation between these forces, which, in turn, increases the amplitudes of the cylinder's oscillations.

A third aeroelastic effect characterizing vortex-induced oscillations is the development of aeroelastic forces associated with flow modifications induced by the oscillations.

The required minimum amplitude levels are established, for example, in [10,11,30], to be around 0.01 D of the mean amplitude or 0.006 D of the standard deviation of lateral displacements. The critical across-wind actions are stochastic processes with a more or less narrow bandwidth, even in turbulent flow, depending mainly on the relative cylinder surface roughness k_r, the wind turbulence intensity I_v, the aspect ratio of the cylinder, and the Reynolds number Re. The bandwidth is narrower in the subcritical and transcritical range, but somewhat extended in the critical and subcritical range of Re due to intermittent instability of the phenomenon. In resonance, the increasing oscillations are found to be limited either by structural damping or by the apparent self-limiting nature of the aeroelastic mechanism itself, which complicates the development of a satisfactory analytical model.

In general, the critical across-wind action caused by vortices on a slender structure with a circular cross-section in the lock-in region can be described by the following formula:

$$w_{yvc} = w_{yvc}(z, y, \dot{y}, \ddot{y}, t) = q_{vc}(z) D(z) \breve{w}_{yvc}(z, y, \dot{y}, \ddot{y}, t) \tag{46}$$

where: z is the along-axis coordinate of the structure; y, $\dot{y}$, $\ddot{y}$ are the lateral displacement, velocity, and acceleration of structural vibration, respectively; $y = y(z,t) \cong \Phi(z)\Psi(t)$, where $\Phi(z)$ is the vibration mode and $\Psi(t)$ is the generalized coordinate; t is time; $q_{vc} =$

$\frac{1}{2}\rho V_{vc}^2$ is the critical velocity pressure; ρ is the air density; $D(z)$ is the structural diameter; $\breve{w}_{yvc}$ is the dimensionless critical across-wind action. It should be noted that, for brevity, only variables regarded as independent (z,t) or dependent $(y,\dot{y},\ddot{y})$ are explicitly presented. The following dimensionless variables, treated as parameters, are not explicitly shown: relative structural surface roughness k_r, Reynolds number Re, fluctuation (turbulence) intensity of the oncoming airflow I_v , Strouhal number St, and slenderness ratio λ.

We can express the general functional relationship for the frequency of vortex shedding f_v as:

$$f_v = f_v(\rho, V, D, \nu, k_r, I_v) \tag{47}$$

where ν is the kinematic viscosity of air.

Assuming the dimensional base to be (ρ, V, D), this relationship can be rewritten in the dimensionless form as:

$$St = \frac{f_v D}{V} = \check{f}_v(Re, k_r, I_v) \tag{48}$$

Taking this relationship into account, it can be stated that formula (46) is, all things considered, a very complex relation. By treating the structure for each vibration mode $\Phi(z)$ and for the generalized coordinate $\Psi(t)$ as a one-degree-of-freedom system, a differential equation for the lateral vibration of a mechanically linear structure can be written in the following general form:

$$M^{\Phi}\left(\ddot{\Psi} + 2\gamma_y\omega_y\dot{\Psi} + \omega_y^2\Psi\right) = W^{\Phi}\left(\Psi,\dot{\Psi},\ddot{\Psi},t\right) = q_{ref}D_{ref}H\breve{W}^{\Phi}\left(\Psi,\dot{\Psi},\ddot{\Psi},t\right) \tag{49}$$

where $M^{\Phi} = \int_o^H [\Phi(z)]^2\, m(z)\mathrm{d}z$ is the generalized mass, $m(z)$ is the mass per unit length of the structure, H is the structural length (height), γ_y is the critical damping ratio ($\gamma_y \cong \Delta_y/(2\pi)$, where Δ is the logarithmic decrement of vibration damping), $\omega_y = 2\pi f_y$; $q_{ref} = q_{vc,ref}$ is the reference velocity pressure, D_{ref} is the reference diameter, W^{Φ} and $\breve{W}^{\Phi}$ are the dimensional and dimensionless generalized critical across-wind actions, respectively, with the actions related by the following relationships:

$$W^{\Phi}\left(\Psi,\dot{\Psi},\ddot{\Psi},t\right) = \int_0^H \Phi(z)\, w_{yvc}(z,y,\dot{y},\ddot{y},t)\mathrm{d}z =$$

$$= q_{ref}D_{ref}H\left(\frac{1}{H}\int_o^H \Phi(z)\,\frac{q_{vc}(z)D(z)}{q_{ref}D_{ref}}\breve{w}_{yvc}\left(z,\Phi\Psi,\Phi\dot{\Psi},\Phi\ddot{\Psi},t\right)\mathrm{d}z\right) =$$

$$= q_{ref}D_{ref}H\breve{W}^{\Phi}\left(\Psi,\dot{\Psi},\ddot{\Psi},t\right) \tag{50}$$

The dimensional and dimensionless critical across-wind actions w and $\breve{w}$ are, in general, space-time stochastic (random) processes of a very narrow-band nature in the subcritical and transcritical range of the Reynolds number Re (i.e., range a), or of a more or less narrow-band nature in the critical and supercritical range of Re (i.e., range b). This also applies to the stochastic processes W^{Φ} and $\breve{W}^{\Phi}$.

The complexity of the feedback phenomenon discussed means that, so far, no satisfactory and complete analytical models have been derived from the fundamental laws governing airflow around a vibrating cylinder. However, there are several semi-empirical models in which the parameters involved must be determined experimentally. For example:

- Harmonic model with additional linear terms, including aerodynamic damping force (dependent on $\dot{y}$) and, very rarely, aerodynamic stiffness force (dependent on y) and aerodynamic inertia force (dependent on $\ddot{y}$) (e.g., Scanlan model [26]; Davenport model, Vickery model [32]);
- Nonlinear models of autonomous (self-excited) systems (e.g., Scanlan model [26], Hartlen and Currie model [16]);
- Wake-oscillator model [28,29];
- Blevins and Burton model [2,3];
- Vickery and Basu model [1,30,31];
- Correlation length model [22,23];
- Nonlinear generalized harmonic models, such as nonlinear amplitude-dependent self-limiting models of the lock-in phenomenon at vortex excitation, developed by A. Flaga [10,11].

The simplest case of the last models will be considered in more detail further. Let us assume that:

(a) the airflow is steady (s) and uniform (i.e. $V = const.$); (b) an undeformable cylinder, elastically supported (including damping) at its ends, represents the physical model of the system (i.e., a sectional model of a structure) (Fig. 4); (c) vortex shedding is periodic (i.e., it occurs in the subcritical or transcritical range of the Reynolds number, i.e., range a).

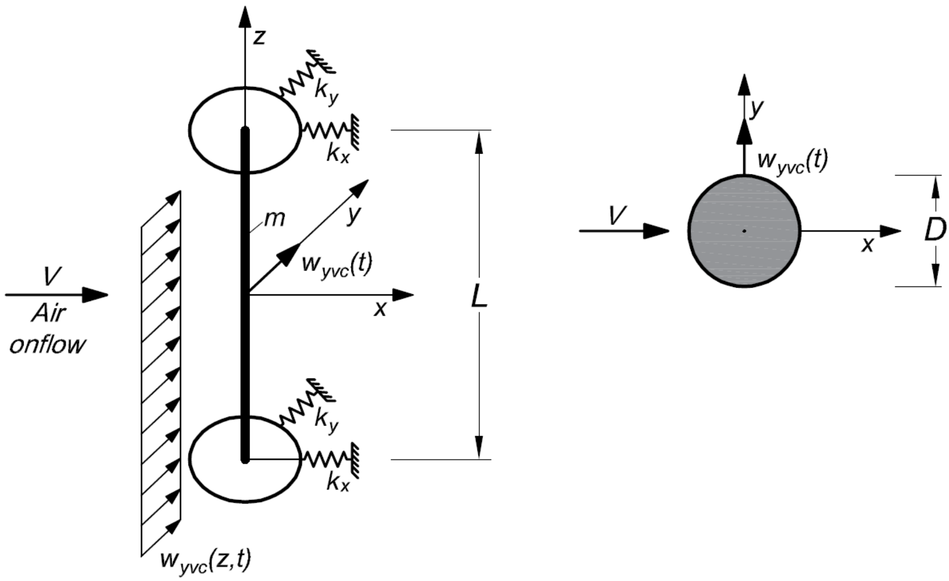

Fig. 4. Sectional model of a slender cylindrical structure under vortex-induced excitation (*source:* authors)

Due to the cross-flow velocity of the vibrating cylinder, both the magnitude and direction of the approaching airflow velocity change periodically. As a result, the position of the separation points on the cylinder's boundary layer may also change periodically. Additionally, the movement of the cylinder boundary causes an apparent increase in the effective diameter – i.e., the characteristic transverse dimension of the vibrating cylinder. Thus, in the dimensional analysis of the vortex-shedding phenomenon, it is necessary to consider the width of the vortex street *B*, the frequency of vortex shedding f_v, and the velocity of vortex displacement V_v, which lead to an increase in the effective diameter compared to the actual cylinder diameter *D*. This increase should also be directly related to the width of the vortex street (i.e. B^{sav} in this context). The effective transverse dimension of the vibrating cylinder D^{sa} can then be described by the following relationships:

$$\frac{D^{sa}}{D} = \frac{B^{sav}}{B^{sao}} = 1 + \alpha^{sa}\frac{Y}{D} = 1 + \alpha^{sa}\breve{Y} \tag{51}$$

where B^{sav}, B^{sao} are the widths of the vortex street for a vibrating and non-vibrating cylinder, respectively; $\alpha^{sa} = \alpha^{sa}$ (*Re*) is a parameter determined experimentally; and Y, $\breve{Y}$ are the dimensional and dimensionless amplitudes of vibration, respectively.

The investigation results by Griffin and others [13-15], conducted in the subcritical range of the Reynolds number, provide, for example, the following result:

$$\frac{B^{sav}}{B^{sao}} = \frac{D^{sa}}{D} \cong 1 + 0.70\breve{Y};\ \propto^{sa} = 0.70 \tag{52}$$

Assuming the effective cylinder dimension D^{sa}, rather than its actual dimension *D*, as the basis for consideration leads to a different value for the critical velocity V_{vc}^{sav}:

$$V_{vc}^{sav} = \frac{f_y D^{sa}}{St} \neq V_{vc}^{sao} = \frac{f_y D}{St},\ \frac{V_{vc}^{sav}}{V_{vc}^{sao}} = \frac{D^{sa}}{D} = 1 + \alpha^{sa}\breve{Y} \tag{53}$$

The occurrence of an extremum in the transverse force due to vortices at the critical velocity $V_{vc}^{sav} > V_{vc}^{sao}$ was confirmed by investigations (e.g., [9]), where $V_{vc}^{sav} \in (1.2 - 1.35)V_{vc}^{sao}$.

Since the ratio $V_{vc}^{sav}/V_{vc}^{sao}$ is bounded, the amplitude of vibration is also limited. For instance, if one assumes $V_{vc}^{sav}/V_{vc}^{sao} = 1.3, \alpha^{sa} = 0.7$, then:

$$\breve{Y} = (V_{vc}^{sav}/V_{vc}^{sao} - 1)/\alpha^{sa} = 3/7 \cong 0.43\ \ (i.e.\, \mathrm{Y}_{\max} \cong 0.43D) \tag{54}$$

Taking into account the effective transverse dimension of the cylinder D^{sa}, a mathematical model for the critical across-wind load w_{yvc}^{sav} (*Y*, *t*) can be written in the form:

$$w_{yvc}^{sav}(Y,t) = q_c^{sav} D^{sa} C^{sao} \sin\left(2\pi f_y t + \varphi^{sav}\right) =$$

$$q_c^{sao} D (1 + \alpha^{sa}\breve{Y})^3 C^{sao} \sin\left(2\pi f_y t + \varphi^{sav}\right) = q_c^{sao} D \breve{w}_{yvc}^{sav}(\breve{Y},\breve{t}) \tag{55}$$

where: $q_c^{sao} = \frac{1}{2}\rho(V_{vc}^{sao})^2$; $q_c^{sav} = \frac{1}{2}\rho(V_{vc}^{sav})^2$; $C^{sao} = C^{sao}(Re)$ – the aerodynamic coefficient for a motionless cylinder; φ^{sav} the phase shift angle.

For the analyzed physical model of the system, the dimensionless amplitude of vibration is given by:

$$\breve{Y} = \frac{q_c^{sao}\, c^{sao}}{2\gamma_y m\omega_y^2}\,(1+\alpha^{sa}\breve{Y})^3, \frac{\breve{Y}}{(1+\alpha^{sa}\breve{Y})^3} = \frac{q_c^{sao}\, c^{sao}}{2\gamma_y m\omega_y^2} = \frac{M\rho c^{sao}}{8\pi^2\gamma_y St^2} \tag{56}$$

This is a non-linear equation for determining $\breve{Y}$. The function $\breve{Y}/(1+\alpha^{sa}\breve{Y})^3$ (Fig. 5) has an extremum at the value $0.148/\alpha^{sa}$ for $\breve{Y} = 0.5/\alpha^{sa}$. Thus, $Y < Y_{max} = 0.5D/\alpha^{sa}$. Further analysis, supported by experimental results, confirms that in the adopted model of across-wind action, the feedback between cylinder vibration and vortex shedding is self-limiting in nature.

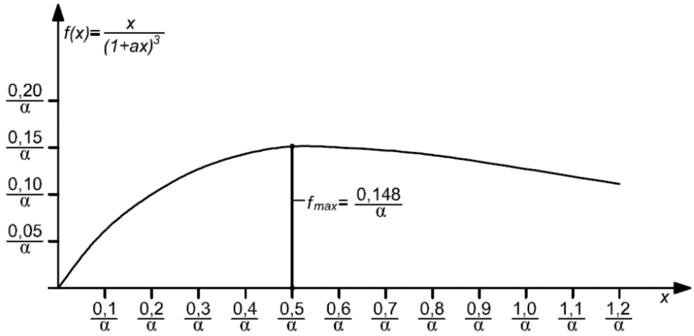

Fig. 5. Plot of the function $f(x) = \frac{x}{(1+ax)^3}$ (*source:* authors)

6. Buffeting wind actions on slender structures

In the design of many slender buildings and structures, wind-structure interactions (i.e., aeroelastic forces and force moment) can often be neglected, with buffeting wind actions caused by oncoming turbulent wind playing the most significant role. Buffeting action components are typically described using so-called quasi-steady models, which are commonly applied in structural aerodynamics (e.g., [6-8]).

For instance, in the case of slender, tower-shaped structures with a longitudinal axis *z*, the buffeting actions per unit length can be expressed as follows:

- wind velocity vector components

$$\vec{V}(z,t) = (V(z) + u(z,t); v(z,t); w(z,t)) \tag{57}$$

- dimensional component of mean wind action

$$\overline{w}(z) = \frac{1}{2}\rho V^2(z) D(z) \tag{58}$$

- along-wind action (i.e. aerodynamic drag)

$$w_{xb} = \overline{w}\,(C_x + 2C_x\frac{u}{V} + C_{xy}\frac{v}{V}) \tag{59}$$

- across-wind action

$$w_{yb} = \overline{w}(C_y + 2C_y\frac{u}{V} + C_{yx}\frac{v}{V}) \tag{60}$$

- torsional wind action

$$w_{nb} = \overline{w}D(C_m + 2C_m \frac{u}{V} + C_{mm} \frac{v}{V}) \tag{61}$$

where:

$$C_{xy} = \left. \frac{\partial C_D}{\partial \beta} \right|_{\beta=0} - C_L \tag{62}$$

$$C_{yx} = \left. \frac{\partial C_L}{\partial \beta} \right|_{\beta=0} + C_D \tag{63}$$

$$C_{mm} = \left. \frac{\partial C_m}{\partial \beta} \right|_{\beta=0} \tag{64}$$

In the case of a slender bridge span, the corresponding relationships can be expressed as follows:

$$w_{xb} = \overline{w}(C_x + 2C_x \frac{u}{V} + C_{xz} \frac{w}{V}) \tag{65}$$

$$w_{zb} = \overline{w}(C_z + 2C_z \frac{u}{V} + C_{zx} \frac{w}{V}) \tag{66}$$

$$w_{mb} = \overline{w}D \left(C_m + 2C_m \frac{u}{V} + C_{mm} \frac{w}{V}\right) \tag{67}$$

Let us consider, as an example, only the along-wind action of turbulent wind on a slender, tower-shaped structure (Fig. 6). Assume it is a rectangular prism with height *H*, width *B*, and thickness *D*. We will consider the building's response as:

$$\xi(z,t) = \sum_i \breve{\Phi}_i^x(z)\Psi_i^x(t) \tag{68}$$

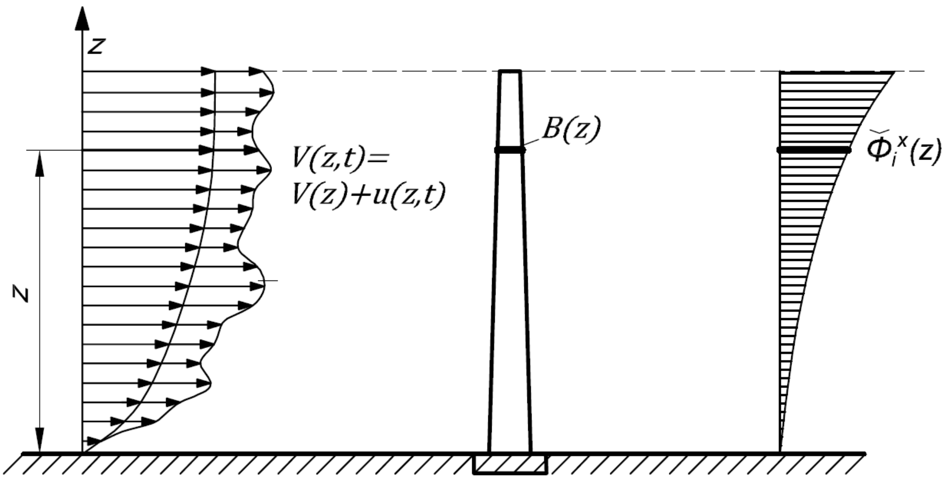

Fig. 6. A tower-shaped structure subjected to buffeting wind action (*source:* authors)

where: $\breve{\Phi}_i^x(z)$ – i-th dimensionless form of bending free vibrations (eigenmode) in the x-direction; z – height above the ground, $\Psi_i^x(t)$ – i-th generalized (main, modal) coordinate in the x-direction. Moreover, we assume that the mean wind direction is perpendicular to the wall with dimensions D x H. Given suitable input assumptions, the motion equation of the structure, corresponding to the first vibration mode $\breve{\Phi}_x(z) = \breve{\Phi}_1^x(z)$, can be approximated as [9]:

$$\frac{d^2\Psi_x(t)}{dt^2} + 2(2\pi f_x)\gamma_x \frac{d\Psi_x(t)}{dt} + (2\pi f_x)^2\Psi_x(t) = \frac{1}{(\rho_b DBH)M_x}\left(\frac{1}{2}\rho C_x DHV_{ref}^2\right)W_x(t) \quad (69)$$

where: $\Psi_x(t) = \Psi_1^x(t)$; f_x – fundamental free vibration frequency, γ_x – critical damping ratio; ρ_b – mean mass density of the structure per unit volume; ρ – air mass density; V_{ref} – mean wind velocity at the reference height; M_x– dimensionless generalized mass given by:

$$M_x = \int_0^1 \left[\breve{\Phi}_x(\zeta)\right]^2 d\zeta \quad (70)$$

$\zeta = z/H; W_x(t)$ – dimensionless generalized force given by:

$$W_x(t) \cong \int_0^1 \left[\frac{V^2(\zeta)}{V_{ref}^2} + 2\frac{u(\zeta,t)}{V_{ref}}\right]\breve{\Phi}_x(\zeta)d\zeta \quad (71)$$

$V(\zeta)$ – mean wind velocity, $u(\zeta, t)$ - wind velocity fluctuations in the x-direction.

Now, we transform the motion equation (69) into a dimensionless form, taking into account the following substitutions and dependencies:

$$\breve{t} = f_o t = \frac{1}{t_o}t; \qquad dt = \frac{1}{f_o}d\breve{t}; \qquad \Psi_x(t) = \frac{\Psi_x\left(\frac{\breve{t}}{f_o}\right)}{D}D = \breve{\Psi}_x(\breve{t})D \quad (72)$$

$$\frac{d\Psi_x(t)}{dt} = f_o D\frac{d\breve{\Psi}_x(\breve{t})}{d\breve{t}}; \qquad \frac{d^2\Psi_x(t)}{dt^2} = f_o{}^2 D\frac{d^2\breve{\Psi}_x(\breve{t})}{d\breve{t}^2}; \qquad q_o = ½\rho V_{ref}^2;\ m = \rho_b DB \quad (73)$$

where the scaling parameter f_o for the dimensionless frequency $\breve{f}$(or $t_o = 1/f_o$ for the dimensionless time $\breve{t}$) is defined as:

$$f_o = \frac{1}{t_o} = \begin{cases} \frac{V_{ref}}{D} \\ f_x \end{cases} \quad (74)$$

and m is the mean mass density per unit height of the structure.

After dividing both sides of eq. (69) by $f_o^2 D$, we obtain:

$$\frac{d^2\breve{\Psi}_x(\breve{t})}{d\breve{t}^2} + 2(2\pi f_x)\gamma_x\left(\frac{f_x}{f_o}\right)\frac{d\breve{\Psi}_x(\breve{t})}{d\breve{t}} + 4\pi^2\left(\frac{f_x}{f_o}\right)^2\breve{\Psi}(\breve{t}) = \frac{q_o}{mf_o^2}C_x\frac{\breve{W}_x(\breve{t})}{M_x} \quad (75)$$

Depending on the assumed reference frequency quantities f_o or V_{ref}/D or f_x, we will proceed accordingly:

$$\frac{d^2\breve{\Psi}_x(\check{t})}{d\check{t}^2}+2(2\pi\gamma_x)Sr_x\frac{d\breve{\Psi}_x(\check{t})}{d\check{t}}+4\pi^2Sr_x^2\breve{\Psi}(\check{t})=\frac{q_oD^2}{mV_{ref}^2}C_x\frac{\breve{W}_x(\check{t})}{M_x}=M\rho_xC_x\frac{\breve{W}_x(\check{t})}{M_x} \tag{76}$$

or

$$\frac{d^2\breve{\Psi}_x(\check{t})}{d\check{t}^2}+2(2\pi\gamma_x)\frac{d\breve{\Psi}_x(\check{t})}{d\check{t}}+4\pi^2\breve{\Psi}(\check{t})=\frac{q_o}{mf_x^2}C_x\frac{\breve{W}_x(\check{t})}{M_x}=M\rho_xVr_x^2C_x\frac{\breve{W}_x(\check{t})}{M_x} \tag{77}$$

where:

$$M\rho_x=\frac{\rho D^2}{2m};\quad Sr_x=\frac{f_xD}{V_{ref}};\quad Vr_x=\frac{V_{ref}}{f_xD} \tag{78}$$

Dimensionless monomials $M\rho_xC_x/M_x$ or $M\rho_xVr_x^2C_x/M_x$ could be treated as a measure of the ratio of aerodynamic force to inertial force. Moreover, another ratio, namely:

$$\frac{\frac{\rho_o}{mf_o^2}}{2\pi\gamma_x\frac{f_x}{f_o}}=\frac{\rho V_{ref}^2}{2m\Delta_xf_xf_o}=\frac{\rho D^2}{2m\Delta_x}\cdot\frac{V_{ref}^2}{f_xf_oD^2}=\frac{M\rho_x}{\Delta_x}\cdot\frac{V_{ref}^2}{f_xf_oD^2}=\frac{1}{Sc_x}\cdot\begin{cases}Vr_x;\ f_o=\frac{V_{ref}}{D}\\ Vr_x^2;\ f_o=f_x\end{cases} \tag{79}$$

where: $Sc_x=\frac{\Delta_x}{M\rho_x}=\frac{2m\Delta_x}{\rho D^2}$ is a new, convenient dimensionless parameter for model investigations, called the Scruton number, which combines damping and mass effects.

Similar considerations can be applied in the case of torsional response $\varepsilon(z,t)$ caused by the aerodynamic moment with the aerodynamic moment coefficient C_m. By formally replacing the index x, with ε, the respective relationships can be expressed as follows:

$$\varepsilon(z,t)=\sum_r\breve{\Phi}_i^\varepsilon(z)\,\Psi_i^\varepsilon(t) \tag{80}$$

$$\frac{d^2\breve{\Psi}_\varepsilon(\check{t})}{d\check{t}^2}+2(2\pi\gamma_\varepsilon)\frac{f_\varepsilon}{f_o}\frac{d\breve{\Psi}_\varepsilon(\check{t})}{d\check{t}}+4\pi^2\left(\frac{f_\varepsilon}{f_o}\right)^2\breve{\Psi}_\varepsilon(\check{t})=\frac{q_oD^2}{m_\varepsilon f_o^2}C_m\frac{\breve{W}_\varepsilon(\check{t})}{M_\varepsilon} \tag{81}$$

$$Vr_\varepsilon=\frac{1}{Sr_\varepsilon}=\frac{V_{ref}}{f_\varepsilon D};\qquad\frac{q_oD^2}{m_\varepsilon f_o^2}=\begin{cases}M\rho_\varepsilon=\frac{\rho D^4}{2m_\varepsilon};\ f_o=\frac{V_{ref}}{D}\\ M\rho_\varepsilon\cdot Vr_\varepsilon^2=\frac{\rho D^4}{2m_\varepsilon}\cdot\left(\frac{V_{ref}}{f_\varepsilon D}\right)^2;\ f_o=f_\varepsilon\end{cases} \tag{82}$$

$$2\pi\gamma_\varepsilon\cong\Delta_\varepsilon;\qquad Sc_\varepsilon=\frac{\Delta_\varepsilon}{M\rho_\varepsilon}=\frac{2m_\varepsilon\Delta_\varepsilon}{\rho D^4} \tag{83}$$

7. Aerodynamic vibrations of a complex structure susceptible to dynamic wind action: an example of cable-stayed or suspended bridge

In the case of complex structures susceptible to dynamic wind action, especially when interference phenomena occur, reliable models of wind action on such structures are typically unknown. In these situations, model tests in wind tunnels become indispensable, and the corresponding similarity criteria must be determined and fulfilled. Let us examine this scenario more closely using the example of cable-stayed or suspended bridges [9]. The following assumptions are to be accepted:

A physical/mathematical model of the structure is defined in accordance with the procedures of the finite element method.

The structure's response, expressed in generalized displacements (i.e., linear displacements or rotational angles), can be described by:

$$\mathbf{A}(t) = \sum_i \boldsymbol{\Phi}_i \psi_i(t) \tag{84}$$

where: $\boldsymbol{\Phi}_i$ – i-th mode of free vibrations; Ψ_i – i-th dimensionless generalized (modal) coordinate. In general, it may be assumed, that

$$\psi_i(t) = \psi_i(\{W\},\{G\},\{O\};t) \tag{85}$$

where: $\{W\},\{G\}$ – sets of dimensional or dimensionless parameters characterizing the oncoming airflow and geometrical features of the object; $\{O\}$ – sets of dimensional or dimensionless parameters characterizing the mechanical properties of the object (i.e., inertia, damping, stiffness).

Within the mechanical parameters of the object $\{O\}$, for example, the following groups can be distinguished:

- case 1

$$\{O\} = \{M_n, M_b, M_m, C_n, C_b, C_m, K_n, K_b, K_m\}; \{\boldsymbol{\Phi}_i\}\} \tag{86}$$

i.e., a set of generalized (modal) quantities: generalized mass M_k, generalized damping C_k, and generalized stiffness K_k; where $k = n, b, m$; denote the respective wind action components (i.e., aerodynamic drag, lift, and moment); and a set of free vibration modes (shapes) $\{\Phi_i\}$;

- case 2

$$\{O\} = \left\{ \begin{Bmatrix} (m, m_m)_{span} \\ (m, m_m)_{pylon} \\ (m)_{cable} \end{Bmatrix}; \gamma_n, \gamma_b, \gamma_m; f_n, f_b, f_m; \{\boldsymbol{\Phi}_i\} \right\} \tag{87}$$

- case 3

$$\{O\} = \left\{ \begin{Bmatrix} (m, m_m)_{span} \\ (m, m_m)_{pylon} \\ (m)_{cable} \end{Bmatrix}; \gamma_n, \gamma_b, \gamma_m; \begin{Bmatrix} \{EI_n, EI_b, GI_m\}_{span} \\ \{EI_n, EI_b, GI_m\}_{pylon} \\ \{EA\}_{cable} \end{Bmatrix} \right\} \tag{88}$$

where: m, m_m – mass and mass moment of inertia per unit length of the main structural elements; γ_k – critical damping ratio (dimensionless quantity); EI, GI_m, EA – bending, torsional, and along-axis stiffness of the main structural elements, respectively.

Assuming a dimensional base of (ρ, D, V), from the dimensional quantities m, m_m, f_k, EI, GI_m, EA, the following dimensionless quantities can be created:

- dimensionless parameters of mass and mass moment of inertia:

$$M\rho = \frac{\rho D^2}{2m}, \qquad M\rho^m = \frac{\rho D^4}{2m_m} \tag{89}$$

- kinematic Strouhal numbers:

$$Sr_k = \frac{f_k D}{V} = \frac{1}{Vr_k}, \quad k = n, b, m \tag{90}$$

- Cauchy numbers:

$$Ca^{EI} = \frac{EI}{\rho V^2 D^4}, \qquad Ca^{GI_m} = \frac{GI_m}{\rho V^2 D^4}, \qquad Ca^{EA} = \frac{EA}{\rho V^2 D^2} \tag{91}$$

The dimensionless quantities presented above form part of the dynamic similarity criteria for the analysed problem.

By grouping all dimensionless quantities characterizing the mechanical properties of the object into a single set $(\check{O})$ and applying the Π theorem of dimensional analysis, the following final relationship can be written:

$$\psi_i(\check{t}) = \psi_i\{\{\check{W}\}, \{\check{G}\}, \{\check{O}\}; \check{t}\} \tag{92}$$

where: $\check{t}$ – dimensionless time assumed as $\check{t} = \frac{V}{D}t$.

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