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ON PRECISION ACOUSTIC WAVE CALCULATION IN A FREQUENCY DOMAIN

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Abstract. *The accuracy of the calculation of acoustic problems formulated in the frequency domain is presented in this work. The issues of the acoustic point sources modelling were discussed and the influence of frequency as well as the impact of the geometry of the analysed area on the accuracy of calculations were indicated. Speaking about the influence of geometry, we mean not only discretization but also the configuration of the considered area, such as for example point sources localization close to the outer edge.*

Keywords: acoustics wave propagation, computational modeling, source simulation, BEM simulation

O PRECYZYJNYM OBLICZANIU FAL AKUSTYCZNYCH W DZIEDZINIE CZĘSTOTLIWOŚCI

Streszczenie. *Dokładności obliczeń zagadnień akustycznych sformułowanych w dziedzinie częstotliwości została przedstawiona w tej pracy. Omówiono problemy modelowania źródeł punktowych oraz wskazano na wpływ częstotliwości a także wpływ geometrii analizowanego obszaru na dokładność obliczeń. Mówiąc o wpływie geometrii mamy na myśli nie tylko dyskretyzacje, ale także konfigurację rozpatrywanego obszaru jak na przykład punktowe źródła energii położone blisko zewnętrznego brzegu.*

Słowa kluczowe: propagacja fal akustycznych, modelowanie obliczeniowe, symulacja źródeł, symulacja metodą elementów brzegowych (MEB)

Introduction

According to [7, 8] four different computational methods engage in acoustic analysis and solutions of the inverse problems: ray tracing, FEM, BEM, and DG-FEM (Discontinuous Galerkin – Finite Element Method). The key problem of each inverse problem is the forward problem, and this paper is devoted accuracy and effectiveness of the calculation of the forward problem for acoustic.

We now briefly introduce several types of problems which frequently occur in practical applications. Problems of wave propagation phenomena are usually classified as interior or exterior, depending on whether one is interested in the sound field in bounded or unbounded regions in space. In some cases, also could be defined the third type of the acoustic problem when the domain of interest is not a simply connected one (see for example Fig. 1c).

Three different types of problem could be formulated for acoustics [2]. Those are:

1) interior problem,
2) exterior problem,
3) hybrid interior-exterior problem.

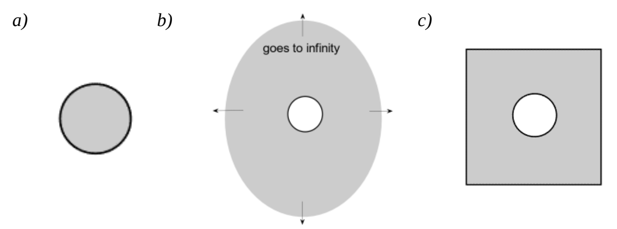

Fig. 1. Sketch of the area for: a) interior problem, b) exterior problem, c) hybrid interior-exterior problem. The greyish area represents the calculation domain

In this paper we would like to focus readers attention on the BEM for interior problem (Fig. 1a) formulated for the frequency domain.

Dedicated iterative methods make it possible to formulate the inverse problems and solve the tomography tasks for acoustic. The advantages of acoustic or ultrasound approach for imaging is obvious and do not demand further explanations.

Ultrasound tomography models are different from the mathematical models formulated for X-ray tomography models [3]. Unlike X-rays, ultrasound waves do not travel in a simple straight line, it undergoes multiple deflections too. The ultrasound wave propagation speed is low, such that delay in propagation times can also be measured. Various methods have been suggested to deal with these refractive problems.

1. Governing equations for the forward internal acoustic problem

The acoustic field is assumed to be present in the domain of a homogeneous isotropic fluid and it is modelled by the linear wave equation [3]:

$$\nabla^2\psi(\mathbf{p},t)=\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\psi(\mathbf{p},t) \tag{1}$$

where $\psi(\mathbf{p},t)$ [m^2/s] is the scalar time-dependent velocity potential related to the time-dependent particle velocity $\mathbf{v}(\mathbf{p},t)=\nabla\psi(\mathbf{p},t)$ [m/s] and c [m/s] is the propagation velocity ($\mathbf{p}$ and t are the spatial and time variables in meters and seconds respectively). The time-dependent sound pressure is equal $p(\mathbf{p},t)=-\rho\frac{\partial}{\partial t}\psi(\mathbf{p},t)$ where ρ [kg/m^3] is the density of the acoustic medium.

Transferring from the time domain to the frequency domain the velocity potential ψ can be expressed as follows:

$$\Psi(\mathbf{p},t)=\mathrm{Re}\{\varphi(\mathbf{p})\mathrm{e}^{-\mathrm{i}\omega\mathrm{t}}\}, \tag{2}$$

where: $\omega=2\pi f$ [1/s] and $\varphi(\mathbf{p})$ is the velocity potential amplitude. The substitution of the above expression into the wave equation reduces it to the Helmholtz equation of the form [3]:

$$\nabla^2\varphi(\mathbf{p})+k^2\varphi(\mathbf{p})=Q, \tag{3}$$

where $k^2=\frac{\omega^2}{c^2}$ and is the wave number and the wavelength is equal to [m]. The right-hand side Q stands for the acoustic source. The complex-valued function $\varphi(\mathbf{p})$ possess the magnitude and phase shift.

Acoustic source term from Eq. (3) very often is treated as the Monopole Source which models a point source that radiates sound isotropically. An example of the acoustic source in 2D space might be a cross section of a cylinder with a small radius which alternately expands and contracts [4].

Such approach could be modelled by the Dirichlet boundary conditions of the internal boundary circle which represent the cross section of the source. But if we make use of the monopole source term Q in Eq. (3) may be written as $Q=Q_0\,\delta(\boldsymbol{p}_s)$. The $\delta(\boldsymbol{p}_s)$ means the Delta Dirac function located in the point $\boldsymbol{p}_s$. Such a mathematical model is particularly convenient in integral formulation of the Partial Differential Equations (PDE). It will be shown in the next section of this paper.

2. Mathematical model of the acoustic source based on hybrid interior-exterior problem

Let us consider hybrid interior-exterior acoustic problem as it is presented in the Fig. 1c. Using this idea, we would like to build the mathematical model for the acoustic source placed in the centre of the square region. The internal boundary should be as small as possible in order to simulate the point source inside the region as it is shown in Fig. 2. Mathematical model based on the Helmholtz equation in the frequency domain [5] in the integral form is described by the following equation.

$$c(\boldsymbol{r})\varphi(\boldsymbol{r}) + \int_{\boldsymbol{\Gamma}} \frac{\partial G(|\boldsymbol{r}-\boldsymbol{r}'|)}{\partial n}\varphi(\boldsymbol{r}')d\boldsymbol{\Gamma} =$$
$$= \int_{\boldsymbol{\Gamma}} G(|\boldsymbol{r}-\boldsymbol{r}'|)\frac{\partial \varphi(\boldsymbol{r}')}{\partial n}d\boldsymbol{\Gamma} \tag{4}$$

Internal boundary (Fig. 2b) of the circular shape with Dirichlet boundary conditions simulates the point source located as it is shown in Fig. 2a.

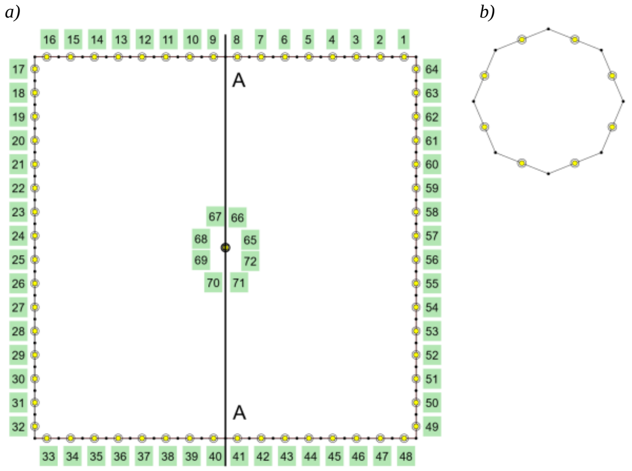

Fig. 2. Discretization of the region under consideration (a) and enlarged internal boundary as a model of the point source (b)

This problem could be treated as an acoustic benchmark for numerical simulation because of its geometrical simplicity. The point sources for the acoustic tomographic problems simulate the multi excitation set so they are especially important. We have to know BEM software behaviour particularly in case of high frequencies like for example ultrasound frequency range.

For one reason the hybrid problem is significant. The unite normal vector have to be directed outside the analysed region. So, for the external boundary the normal derivative would be directed to infinity but for internal boundary to the empty subregion (see for example Fig. 2b).

The following figures shows the solution of the problem with centred point source modelled by hybrid interior – exterior problem. The left-hand side column a) belongs to analytical solution based on Eq. (5), but the right-hand side figures represent the numerical solution. In Eq. (5) the Q_0 is the point source strength but in the BEM model it is replaced by Dirichlet boundary conditions.

Due to singularity of the solution the hybrid model of the problem makes only possible qualitative comparison between analytical and numerical calculations. The singularity point in numerical model is excluded and replaced by Dirichlet boundary conditions imposed on the internal boundary circumfluent the real point source position (see Fig. 3 and Fig. 4).

In the Fig. 3 and Fig. 4 acoustic field is presented for two frequencies: 20 Hz the lowest audible frequency (top row) and 680 Hz frequency (bottom row) [1]. At the external boundary homogeneous Dirichlet boundary conditions were imposed (Sound Soft BC).

It is clear that such approach could be applied to model acoustic point sources particularly useful in acoustic tomography.

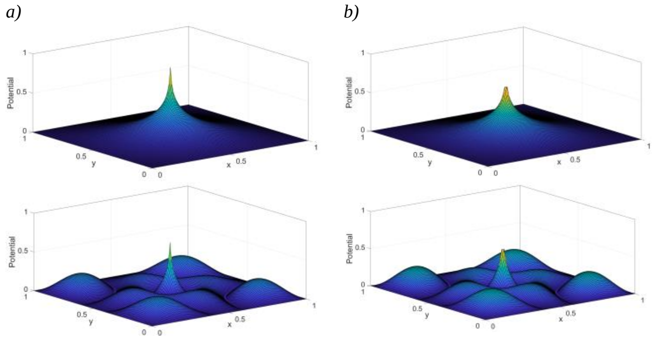

Fig. 3. Relief plot of absolute value of complex potential distribution for the analytical solution - column a) and for BEM numerical solution – column b)

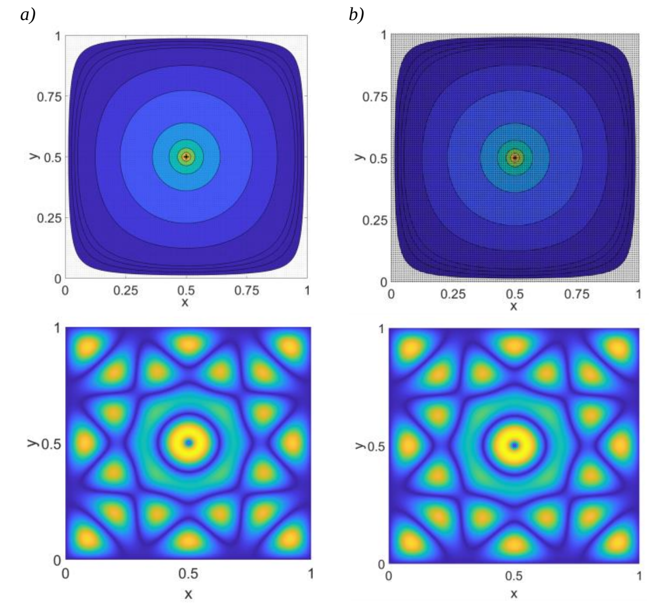

Fig. 4. Qualitative comparison between analytical (left, column a) and BEM (right, column b) solutions for the Helmholtz equation for a source located in the centre of the square region

3. Acoustic source modelled by Delta Dirac function

Similar task but this time with the point source modelled by the Delta Dirac function were considered. In order to make the Quantitative comparison following [1] let us consider the analytical solution of the acoustic problem formulated in previous paragraph:

$$\varphi(\boldsymbol{x},\boldsymbol{y}) = \frac{4Q_0}{ab}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\left[\frac{\sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{m\pi y}{b}\right)\sin\left(\frac{n\pi\eta_0}{a}\right)\sin\left(\frac{m\pi\xi_0}{b}\right)}{\pi^2\left(\frac{n^2}{a^2}+\frac{m^2}{b^2}\right)-k^2}\right] \tag{5}$$

where Q_0 is the point source strength, $a = b$ dimensions of the square region.

Equation (5) describes the outgoing wave of wavenumber k, produced by a point source of strength Q_0, located at point (η_0, ξ_0), observed at (x, y) subject to homogeneous Dirichlet boundary conditions ($\varphi = 0$) on the external boundary of the unite square.

Having the analytical solution such a problem could be treated as a benchmark problem. The issue of precision of the Boundary Element solution with respect to the excitation frequency as well as the spatial discretization would be considered.

The Helmholtz equation should be modified by an integral over the whole region Ω (see the last term of Eq. (6)).

$$c(\boldsymbol{r})\varphi(\boldsymbol{r}) + \int_{\boldsymbol{\Gamma}} \frac{\partial G(|\boldsymbol{r}-\boldsymbol{r}'|)}{\partial n}\varphi(\boldsymbol{r}')d\boldsymbol{\Gamma} =$$
$$= \int_{\boldsymbol{\Gamma}} G(|\boldsymbol{r}-\boldsymbol{r}'|)\frac{\partial \varphi(\boldsymbol{r}')}{\partial n}d\boldsymbol{\Gamma} - \int_{\Omega} G(|\boldsymbol{r}_s-\boldsymbol{r}'|)\,Q_0\delta_s d\Omega \tag{6}$$

where Q_0 is the magnitude of the source and δ_s is a Dirac delta function which integral is equal to one at the point $\boldsymbol{r}_s = \boldsymbol{p}_s$

and zero elsewhere. Taking above into account and assuming that only one point source exists, after some integration Eq. (6) could take the following form:

$$c(\boldsymbol{r})\varphi(\boldsymbol{r}) + \int_{\boldsymbol{\Gamma}} \frac{\partial G(|\boldsymbol{r}-\boldsymbol{r}'|)}{\partial n}\varphi(\boldsymbol{r}')d\boldsymbol{\Gamma} =$$
$$= \int_{\boldsymbol{\Gamma}} G(|\boldsymbol{r}-\boldsymbol{r}'|)\frac{\partial\varphi(\boldsymbol{r}')}{\partial n}d\boldsymbol{\Gamma} - G(|\boldsymbol{r}_s-\boldsymbol{r}'|)Q_s \qquad (7)$$

where for internal points of the region Ω coefficient $c(\boldsymbol{r})=1$.

The results of calculations are presented below. The left-hand column of the Fig. 5a contain the analytical solution and the right-hand side column the BEM solution. Because the solution was achieved in the frequency domain in the figures only the modulus of the complex amplitude is presented.

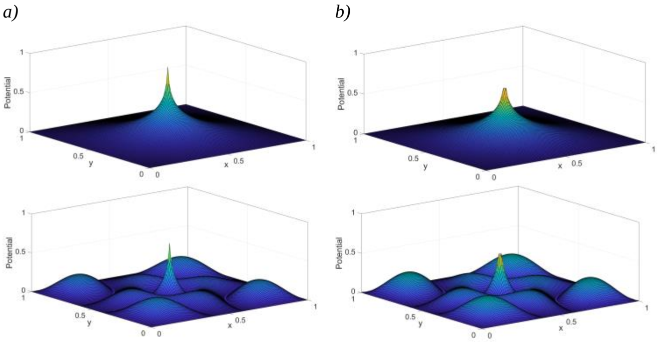

Fig. 5. Relief plot of absolute value of complex potential distribution for the analytical solution – column a) and for BEM numerical solution – column b)

The image of the acoustic field is presented in the Fig. 6 for two frequencies 20 Hz and 680 Hz. Using those figures only qualitative comparison to the benchmark is possible.

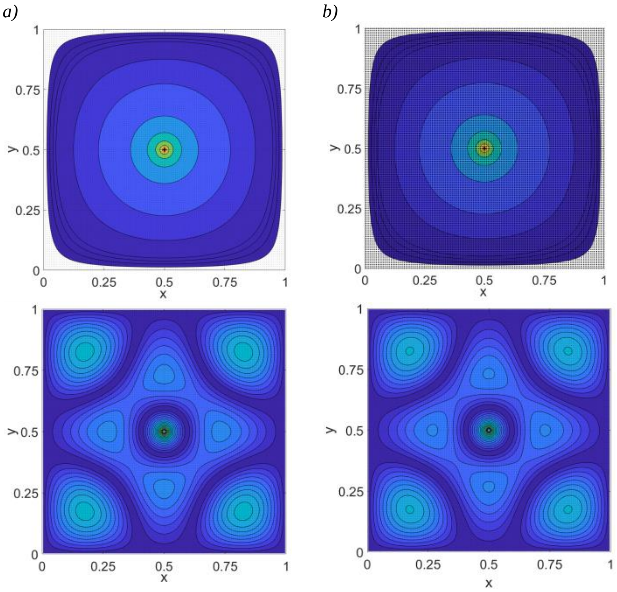

Fig. 6. Qualitative comparison between analytical (left, column a) and BEM (right, column b) solutions for the Helmholtz equation for the point source located in the centre of the square region

If the line A-A in Fig. 2a would be considered than the quantitative comparison between the analytic solution (treated as a benchmark) and the numerical solution became possible.

The boundary element discretization was modest because only sixty-four boundary elements were used. In case of the frequency 20 Hz agreement was excellent but when the frequency become higher the discrepancy become bigger. Therefore, it is necessary to increase the number of boundary elements. We can observe the reduction of a relative error in case of frequency 680 Hz comparing the Fig. 7b and Fig. 9b.

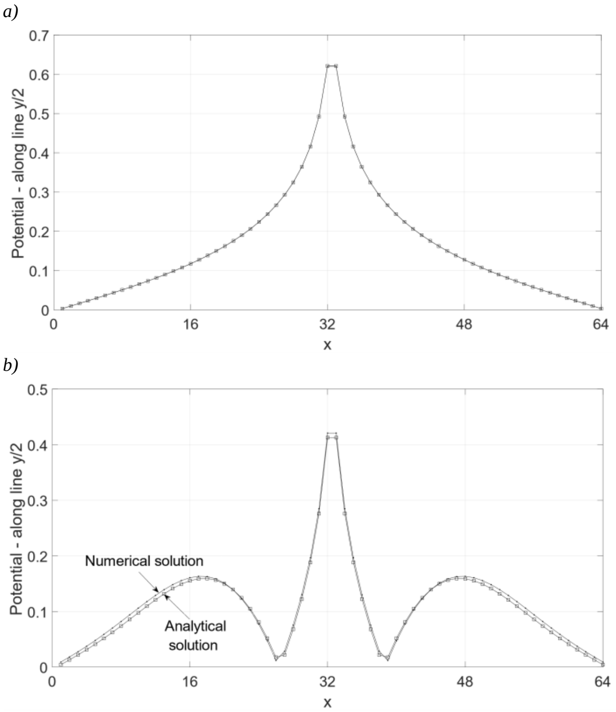

Fig. 7. Potential (acoustic pressure) comparison along the A-A line (see Fig. 2a) for 20 Hz (above) and 680 Hz (below)

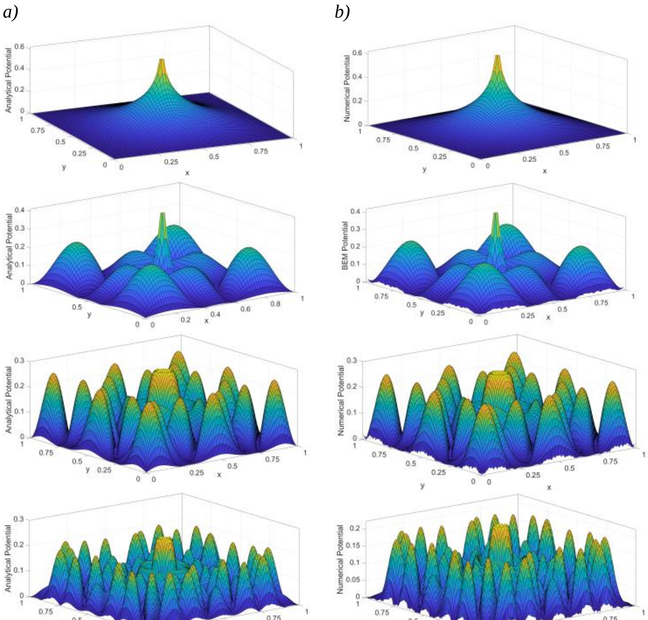

Fig. 8. Qualitative comparison between analytical (left, column a) and BEM (right, column b) solutions of the Helmholtz equation for the point source located in the centre of the square region for the frequencies: 20 Hz, 680 Hz, 1340 Hz and 2000 Hz

Along the A-A line (see Fig. 2a) comparison between analytic and numerical solution is presented below. The highest source frequency the bigger relative error one may observe. The error reduction is possible by increasing the number of boundary elements.

Inspecting those figures one conclusion is clear, that acoustic problem could effectively be solved for a wide range of acoustic parameters of the environment and excitation (frequency for example). The maximal relative error does not exceed 10% and easily could be reduced by applying dense discretization.

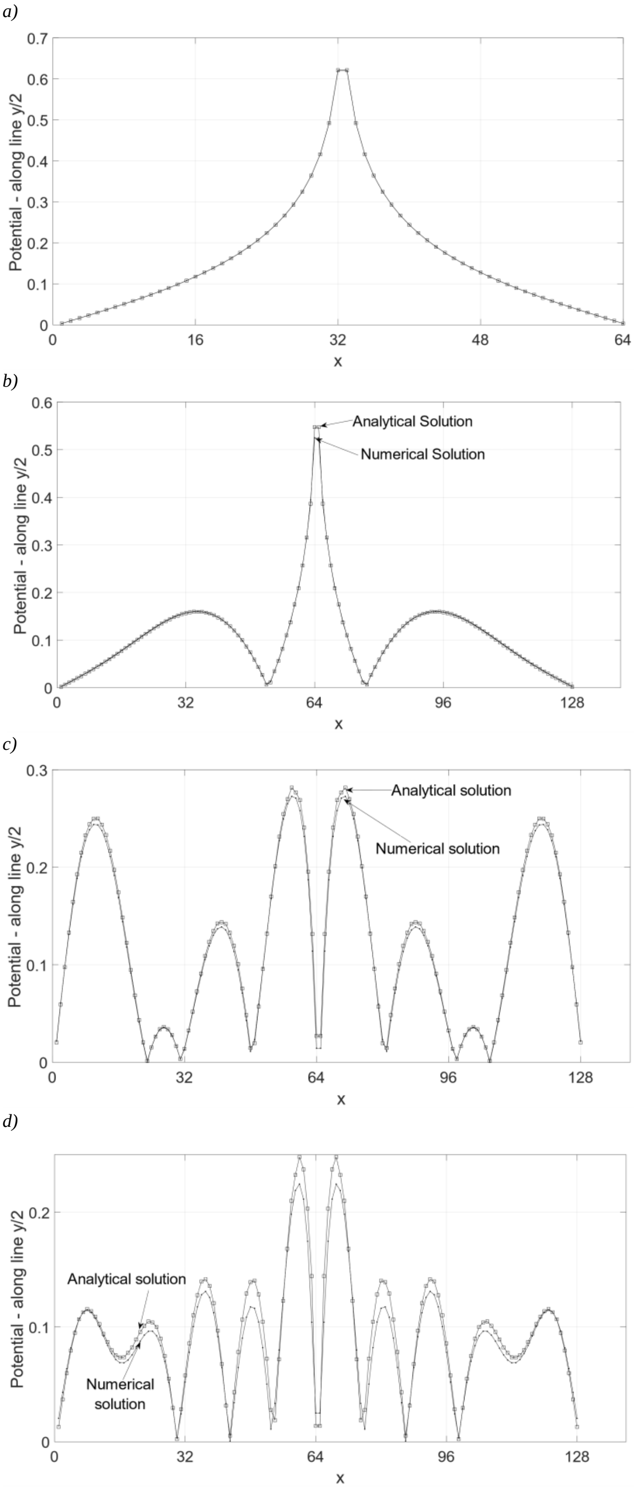

Fig. 9. Potential (acoustic pressure) comparison along the A-A line (see Fig. 2a) for 20 Hz (above) and subsequently 680 Hz, 1340 Hz and finally 2000 Hz

Table 1. Basic data for acoustic benchmark calculation

f [Hz]	20	680	1340	2000
δ%	0	-2.02	3.19	9.48
k [1/m]	0.37	12.42	24.48	36.53
λ [m]	17.20	0.51	0.26	0.17
element length [m]	0.1250	0.0625	0.0625	0.0625
arg. of Henkel function	0.058	1.977	3.895	5.814
no of BE per λ	138	8	4	3

From tomography point of view the most important is a grid providing a minimum number of points per wavelength to resolve acoustic problem even for the highest frequencies. As our goal is the ultrasound tomography, we have to consider frequency above 20000 Hz. For such frequencies, the wavelength became noticeably short even less than 0.017 m.

From tomography point of view satisfactory target would be such a number of boundary elements which allows to achieve a relative error of less than 10%. Furthermore, this selection adheres to the eight point per wavelength rule suggested by [1] for the approximation of acoustic waves. Only frequencies 20 Hz and 680 Hz fulfil this condition. But it does not mean that the remains cases are nor useful for the tomography cases. The relative error remains low even thou the number of elements is about three per wavelength.

However, to calculate acoustic problem of the ultrasound frequency and preserve this condition the region of interest should be much smaller. It means that the acoustic wavelength must be much greater than the length scale of the geometry. So, instead the unite square region the size was reduced by ten up to 0.1 m.

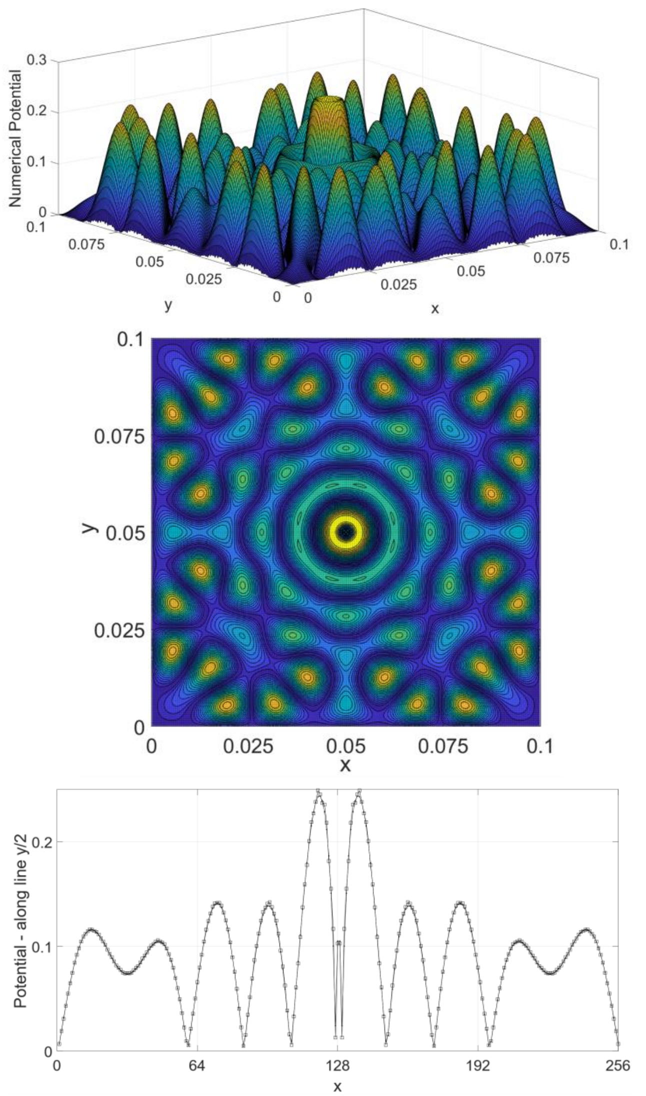

Fig. 10. Ultrasound modelling of the Helmholtz problem with soft boundary conditions

The results are quite satisfactory and the maximal error (with the exception of singularity point in the centre of region) is less than 2.45%.

4. Near-Boundary Source

For tomography problems it is necessary to fix the sensor on the boundary or inside the region but remarkably close to the boundary. It is interesting to investigate numerical solution behaviour in such cases. Let us consider a situation with one point source located near boundary and on the external boundary of unite square the homogeneous Dirichlet boundary conditions were imposed (sound soft boundary conditions).

a) frequency f = 1340 Hz *b) frequency f = 20000 Hz*

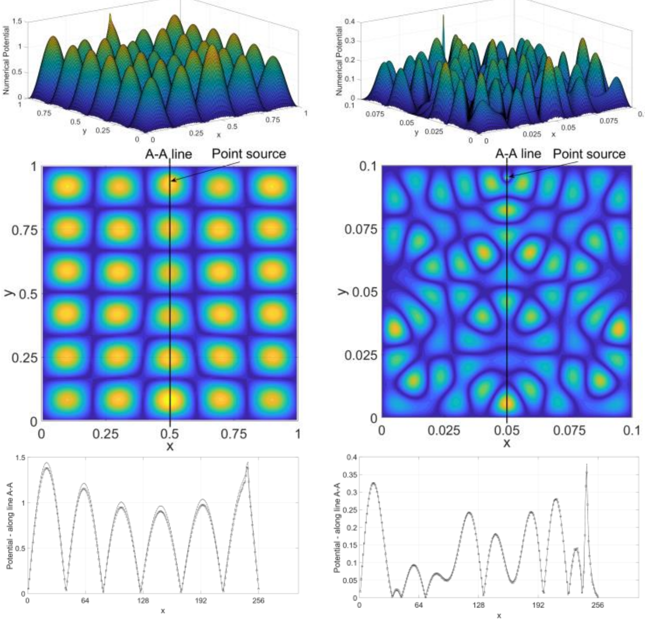

Fig. 11. Ultrasound modelling of the Helmholtz problem with a near-boundary point source and sound soft boundary conditions: column a) for excitation with frequency 1340 Hz, column b) with excitation of 20000 Hz

It is easily to notice that the precision of calculations depends not only on the frequency but also depends on the formulation of the acoustic problem itself. Regarding frequency, the higher frequency the more rigoristic demands of the problem. Particularly it concerns the ultrasound frequencies. Precision might be kept on low level but the ratio of wavelength to the length of boundary element should be not less than eight.

That demand fine discretization what means high time consumption. But tomographic problems which are solved by iteration process have to be as fast in each iteration step as possible. In some particular cases (see table 1) the number of boundary elements per wavelength could be reduce up to 3 or 4 and the maximal relative error would not exceed 10%. It might provide satisfactory results for the tomography imagining.

5. Conclusions

Very often in tomography problems, we have to deal with many sensors emitting and receiving signals which are closely located to the external boundary. As an example, a Diffuse Optical Tomography or Radio Tomography could be mentioned [5, 6].

In this paper two different mathematical models of the acoustic point sources were investigated. For the purposes of the forward tomography problem formulated in the integral form the second mathematical model of the acoustic point source is more convenient.

In all modalities of the tomography only external boundary is accessible so the integral form and the Boundary Element Method possess obvious advantages over the Finite Element Method [5, 6]. The second approach to the acoustic point source involves less boundary elements so it is more convenient to the tomography. Even applying constant boundary elements results of the Helmholtz equation in a broad range of frequency is able to provide results with a maximal relative error less than 10%. In the literature [1–3] is stressed that the acoustic wavelength should be much greater than the length scale of the region under consideration. That simply means that the ratio of the wavelength to the length of the boundary element should be at least equal to eight. Then the precision of calculation would be secured. We can see that in the table 1.

However, in tomography sometimes might be difficult to fulfil such rigorous demands. For example, for the ultrasound frequency band the length of the boundary elements should be extremely small if the level of the error should be kept on the low level. From the point of view of the Inverse Problem efficiency calculation such decision would be difficult to justify. Some compromise between the accuracy and the execution time has to be preserved. We can see that in the case of ultrasound frequency when only three boundary elements within the length of the wave provide results of the forward solution with a maximal error less than 10% inside the region. But on the boundary this error could be even less. In the authors opinion such coarse discretization might be sufficient.

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