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IMPROVING α-PARAMETERIZED DIFFERENTIAL TRANSFORM METHOD WITH DANDELION OPTIMIZER FOR SOLVING ORDINARY DIFFERENTIAL EQUATIONS

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Abstract. *In this manuscript, we aim to address Ordinary Differential Equations (ODEs) by α-Parameterized Differential Transform Method (α-PDTM). Additionally, we seek to enhance the effectiveness of α-PDTM by incorporating the Dandelion Optimizer (DO). The DO plays a crucial role in optimizing the parameter α, ensuring its adjustment and modification to secure the most favorable value. This refinement results in a more accurate approximation compared to conventional methods. The proposed approach, referred to as (α^{DO}-PDTM), demonstrates a solution distinguished by its reliability and efficiency, as determined through the computation of Maximum Absolute Error (MAE) and the Mean Square Errors (MSE).*

Keywords: α-parameterized differential transform, Dandelion optimizer, ordinary differential equations, meta heuristic

UDOSKONALENIE α-PARAMETERYZOWANEJ METODY PRZEKSZTAŁCENIA RÓŻNICZKOWEGO Z OPTYMALIZATOREM DANDELION DO ROZWIĄZYWANIA RÓWNAŃ RÓŻNICZKOWYCH ZWYCZAJNYCH

Streszczenie. *Celem niniejszego manuskryptu jest rozwiązanie równań różniczkowych zwyczajnych (ODE) metodą α-parametryzowanej transformacji różniczkowej (α-PDTM). Ponadto staramy się zwiększyć skuteczność α-PDTM poprzez włączenie optymalizatora Dandelion (DO). DO odgrywa kluczową rolę w optymalizacji parametru α, zapewniając jego dostosowanie i modyfikację w celu zabezpieczenia najbardziej korzystnej wartości. To udoskonalenie skutkuje dokładniejszym przybliżeniem w porównaniu z metodami konwencjonalnymi. Proponowane podejście, określane jako (α^{DO}-PDTM), demonstruje rozwiązanie wyróżniające się niezawodnością i wydajnością, co zostało określone poprzez obliczenie maksymalnego błędu bezwzględnego (MAE) i średnich błędów kwadratowych (MSE).*

Słowa kluczowe: α-parametryzowane przekształcenie różniczkowe, optymalizator Dandelion, równania różniczkowe zwyczajne, metaheurystyka

Introduction

Recently, there has been an increase in interest in differential equations with their broad applications spread across diverse scientific fields and their representation of many phenomena. It is generally hard to find a universal method that provides exact solutions for all differential equations, as only a limited subset can be treated effectively by direct methods. As a result, approximate methods have become prevalent, leading to results that are often considered acceptable and manageable., such as Variational Iteration Method (VIM) [5], Homotopy Analysis Method (HAM) [6], and other methods.

α-PDTM, an extension of the semi-analytical method technique known as the differential transformation method (DTM), was introduced by Zhou in 1986 [14]. This method, a numerical approach, was employed to address both initial and boundary value problems in differential equations, specifically those arising in the analysis of electrical circuits. DTM is used as an alternative to traditional methods for solving differential equations of various types. DTM is based on the idea of converting the differential equation into a set of algebraic equations using the transformation properties that will be mentioned and then solving them to obtain the approximate solution.

The differential transformation of the function $y(x)$ with order k is defined as follows

$$W(k) = \frac{1}{k!}\left[\frac{d^k y(x)}{dx^k}\Big|_{x=0}\right] \tag{1}$$

where $Y(k)$ is Differential Transform (DT) of $y(x)$, and the inverse DT of the is defined as

$$w(x) = \sum_{k=0}^{\infty} W(k)(x-x_0)^k \tag{2}$$

The differential transformation of basic mathematical operations can be illustrated in Table 1 [14].

And the differential inverse transform is defined as

$$w(x) = \sum_{k=0}^{\infty} D_\alpha(f;k)(x-x_0)^k \tag{3}$$

This paper is organized as follows for the remaining sections: In Section 1, the α-PDTM method conditional on the parameter α is explained. Section 2 provides an in-depth explanation of the smart DO algorithm. The section 3 also explains Some important definitions that will be used in this research. All results obtained in this study are covered in Section 4. Finally, in Section 5, the most important general conclusions are drawn.

Table 1. The fundamental operation performed by DTM

Original functions	Transformed functions
$w(x) = g(x) \pm h(x)$	$W(k) = Y(k) \pm H(k)$
$w(x) = cg(x)$	$W(k) = cY(x)$, where c is constant
$w(x) = y'(x)$	$W(k) = (k+1)Y(k+1)$
$w(x) = y^m(x)$	$W(k) = ((k+m)!/k!)Y(k+m)$
$w(x) = \alpha x^n$	$W(k) = \alpha\delta(k-n)$; where $\delta(k-m) = \begin{cases}1 & \text{if } k=n \\ 0 & \text{if } k \neq n\end{cases}$
$w(x) = e^{\alpha y}$	$W(k) = \lambda^k/k!$

1. The general framework of α-PDTM

In this part, we will talk about the operation of the proposed method α-PDTM. Let $-\infty < a < b < \infty$ and $f(a,b) \to R$ be analytic and be $Y_k(f,x_0)$ is the differential transform of f of the $x_0 \in [a,b]$

$$Y_k(f,x_0) = \frac{f(x_0)^k}{k!} \tag{4}$$

$$D(f,\alpha;k) = \alpha Y_k(f,a) + (1-\alpha)Y_k(f,b), k = 0,1,2,\ldots \tag{5}$$

For $\alpha \in [0,1]$, the sequence of $D_\alpha(f)$ is can be defined as

$$D_\alpha(f) = (D(f,\alpha;0), D(f,\alpha;1), D(f,\alpha;2) + \cdots) \tag{6}$$

In these cases $\alpha = 0$ and $\alpha = 1$ we will impose fixed solutions for the values $D(f,\alpha;0) = A$ and $D(f,\alpha;1) = B$, then we calculate the other values of $D(f,\alpha;0)$, $D(f,\alpha;1)$, $D(f,\alpha;2) + \cdots$ from the differential transformation of the differential equation after determining the number of iterations. The final approximate solution to the differential transform with parameters α (α-PDTM) is by calculating the inverse differential transform α, which can be calculated using the following formula [12]:

$$y(x) = \sum_{k=0}^{\infty} D_\alpha(f,\alpha;k)(x-x_\alpha) \tag{7}$$

where $x_\alpha = a\alpha + (1-\alpha)b,$ after substituting different values of x in the defined interval $[a,b]$, we will get an algebraic equation in terms of the variable α, and by using DO we will get the best α value.

2. Dandelion optimizer

2.1. Basic principles

The DO algorithm presented by the scientist Shiji Zhao [13] is considered one of the famous algorithms that outperform many other algorithms in terms of the quality of solutions and speed of convergence [11]. DO is classified as a Meta-Heuristic algorithm [2] that continuously works to find a solution based on Exploration and Exploitation [10], Consideration must be given to two crucial factors. An algorithm can become challenging to solve with excessive exploration, and prioritizing exploitation may result in the convergence of solutions. This shows us the great importance of exploration and exploitation and achieving a balance between them, which confirms the importance of achieving a balance between exploration and exploitation in smart algorithms [1]. The DO algorithm is among those inspired by dandelion seeds, which disperse over long distances with the aid of wind. The growth process of dandelion is segmented into three stages. Planting dandelion seeds is divided into several stages. The first stage involves an upward movement, where the wind helps the seeds spread to remote areas during sunny conditions with wind or drift in cloudy conditions with rain. Then the seeds move to the landing stage, where they gradually decrease after the seed reaches a certain height. Finally, In the next stage specifically, and the seeds settle some where after being affected by weather and wind conditions, starting a new generation of dandelion plants, Based on these three stages, the dandelion optimization algorithm updates individuals' positions. In the bull phase, there are instances of exploration and exploitation, influenced by randomly generated numbers. Furthermore, the adjuster of length is employed to control the fluctuation In seed positions. Dandelion Optimizer depends on two primary parameters,: seed dispersion radius η and local search Coefficient ξ. These parameters change over time continuously in each iteration. Specifically, η determines the global search step length, while ξ dictates the local search step length [11].

2.2. Initialization stage

A population matrix of the identified seed set with N individuals is defined in space d. We represent any seed with the symbol $X_v = [X_i^1, X_i^2, \dots X_i^d]$ wher $v = 1,2,3,\dots$ to initialize the population use this function

$$X_i = lb + r_v(ub - lb) \quad (8)$$

Let r_v denote a randomly generated number according to a normal distribution within the interval $(0,1)$. Here, the highest value of the decision space will be represented by the symbol ub, the lowest value of the decision space will be represented by the symbol lb, N indicates the number of least rows in the population matrix, and d represents the columns of the population matrix. In subsequent sections, these parameters remain with the same symbols [11].

2.3. Upward stage

In the bottom-up phase, we determine what we will do (global search or local exploitation) based on r, which is a random number with a normal distribution

- If $r < 1.5$ if the cut is clear, the following equation is used to determine the new seed location:

$$X_{t+1} = X_t + \eta\, v_x v_y\, ln(Y)\,(X_s - X_t) \quad (9)$$

where X_s is the initial seed position, v_x and v_y represents the two wind directions (horizontal and vertical), η represents the seed dispersal radius, and $ln(Y)$ represents the lognormal distribution with mean 0 and variance 1.

$$X_s = rand(1,d)(ub - lb) + lb \quad (10)$$

$$\eta = rand(\,)(\tfrac{1}{T^2}t^2 - \tfrac{2}{T}t + 1) \quad (11)$$

$$v_x = r\cos(\theta) \quad (12)$$

$$v_y = r\sin(\theta) \quad (13)$$

where $r = \frac{1}{e^{\theta}}$, $\theta \in [-\pi, \pi]$, T represents the total number of iterations, and t is the current iteration.

- If $r \geq 1.5$, it indicates rainy weather, then the next location of the seed is determined

$$X_{t+1} = X_t\, \xi_{\,1} \quad (14)$$

where the local search factor ξ appears in the formula

$$\xi = 1 - rand(\,)\, q \quad (15)$$

$$q = \tfrac{1}{T^2-2T-1}t^2 - \tfrac{2}{T^2-2T-1}t + 1 + \tfrac{1}{T^2-2T-1} \quad (16)$$

2.4. Descending stage

The algorithm descends when it reaches a certain level of height, and through Brownian motion the seeds are in their position determined by the following equation [11].

$$X_{t+1} = X_t - \eta\, \beta_t(X_{meant} - \eta\, \beta_t X_t) \quad (17)$$

$$X_{meant} = \tfrac{1}{n}\textstyle\sum_{i=1}^{n} X_i \quad (18)$$

where β_t is the Brownian motion, X_{meant} is the stability parameter.

2.5. Land stage

In the landing stage, the position of the seeds is as shown in the following formula

$$X_{t+1} = X_{elite} + leve(\lambda)\, \eta\, (X_{elite} - X_t\, \tfrac{2t}{T}) \quad (19)$$

$$leve(\lambda) = s\,(w \times \sigma)/|t|^{1/\beta} \quad (20)$$

$$\sigma = \left(\Gamma(\,1+\beta)\, sin\left(\tfrac{\pi\beta}{2}\right)\right)\left(\Gamma\left(\tfrac{1+\beta}{2}\right)\beta\, 2^{\left(\frac{\beta-1}{2}\right)}\right) \quad (21)$$

where $s = 0.01, \beta = 1.5$, Where the gamma function is defined by the symbol Γ, and w, t are numbers in the interval [0, 1] [11].

3. General concepts

We will discuss some important definitions that we will use in this research.

- Definition (1)
  The maximum absolute errors (MAE) is defined by

$$\|W_{Exact}(y) - T_i(y)\|_\infty = max_{a\le x\le b}\{|W_{Exact}(y) - T_i(y)|\} \quad (22)$$

where $T_i(y)$ is the approximate solution [8].

- Definition (2)
  The mean square errors (MSE) is defined by

$$MSE = \tfrac{1}{n}\,(W_{Exact}(y) - T_i(y))^2 \quad (23)$$

where $T_i(y)$ is the approximate solution [9].

4. The proposed method α^{DO}-PDTM

The proposed method calculates the best value of parameter α in α-PDTM through the process of coupling α-PDTM and DO. We will solve some examples of ODEs by α^{DO}-PDTM and compare the results with Exact Solution.

4.1. Example 1

Consider the following of homogeneous ODE [4]:

$$y''(x) + 2y(x) = 0, x \in [-1,0] \quad (24)$$

with the boundary conditions

$$y(-1) = 0 \text{ and } y'(0) = 1 \quad (25)$$

and the exact solution when

$$y(x) = \tfrac{1}{2}\left(\sqrt{2}\,\sin(\sqrt{2}x) + \sqrt{2}\cos(\sqrt{2}x)\tan(\sqrt{2})\right) \quad (26)$$

If α-PDTM is applied to both sides of (24) we obtain

$$D(y''+2y,\alpha;k)=$$
$$=(k+1)(k+2)D(y,\alpha;k+2)+2D(y,\alpha;k)=0 \quad (27)$$

from the close form of α-PDTM

$$y_\alpha(x)=\sum_{k=0}^{n} D(y,\alpha;k)(x-x_\alpha)^k \quad (28)$$

where n is the number of iterations, $x_\alpha = a\alpha + (1-b)\alpha$ for the boundary conditions $y(-1)=0$ and $y'(0)=1$ we have

$$y_\alpha(-1)=\sum_{k=0}^{n} D(y,\alpha;k)(\alpha-1)^k=0$$
$$y'_\alpha(0)=\sum_{k=0}^{n} kD(y,\alpha;k)(\alpha-1)^{k-1}=1$$

Denote that $D(y,\alpha;0)=A$ and $D(y,\alpha;1)=B$, then find the other values of $D(y,\alpha;k)$ from (27)

$$D(y,\alpha;2)=-A, D(y,\alpha;3)=\frac{-B}{3}, D(y,\alpha;4)=\frac{A}{6}$$
$$D(y,\alpha;5)=\frac{B}{30}, D(y,\alpha;6)=\frac{-A}{90}, \ldots$$

Substitute the previous values of $D(y,\alpha;k)$ into (28) to get the approximate solution

$$y(x,\alpha)=\sum_{k=0}^{6} D(y,\alpha;k)(x+\alpha)^k=$$
$$=A+(x+\alpha)B-(x+\alpha)^2A-(x+\alpha)^3\frac{B}{3}+$$
$$+(x+\alpha)^4\frac{A}{6}+(x+\alpha)^5\frac{B}{30}-(x+\alpha)^6\frac{A}{90} \quad (29)$$

where $x_\alpha=-\alpha$. When we substitute the boundary conditions into (29), we get:

$$y(-1,\alpha)=A+(\alpha-1)B-(\alpha-1)^2A-(\alpha-1)^3\frac{B}{3}+$$
$$+(\alpha-1)^4\frac{A}{6}+(\alpha-1)^5\frac{B}{30}-(\alpha-1)^6\frac{A}{90}=0 \quad (30)$$
$$y'(0,\alpha)=B-2A\alpha-B\alpha^2-\frac{2A}{3}\alpha^3+\frac{B}{6}\alpha^4-\frac{A}{15}\alpha^5=1 \quad (31)$$

When solving the previous (30) and (31), we will obtain the values of A and B

$$A=-\frac{90(-21+5\alpha+20\alpha^2-5\alpha^4+\alpha^5)}{420+30\alpha^2-120\alpha^3+220\alpha^4-216\alpha^5+105\alpha^6+40\alpha^7-30\alpha^8+\alpha^{10}}$$
$$B=-\frac{30(-14-126\alpha+15\alpha^2+40\alpha^3-6\alpha^5+\alpha^6)}{420+30\alpha^2-120\alpha^3+220\alpha^4-216\alpha^5+105\alpha^6+40\alpha^7-30\alpha^8+\alpha^{10}}$$

Substitute the values of A and B into (29), to get:

$$y(x,\alpha)=$$
$$-\frac{90(20\alpha^2-21-5\alpha^4+\alpha^5+5\alpha)}{-120\alpha^3-216\alpha^5+420+30\alpha^2+220\alpha^4+105\alpha^6+40\alpha^7+\alpha^{10}-30\alpha^8}-$$
$$-\frac{30(x+\alpha)(\alpha^6-6\alpha^5+40\alpha^3+15\alpha^2-126\alpha-14)}{-120\alpha^3-216\alpha^5+420+30\alpha^2+220\alpha^4+105\alpha^6+40\alpha^7+\alpha^{10}-30\alpha^8}+$$
$$+\frac{90(x+\alpha)^2(20\alpha^2-21-5\alpha^4+\alpha^5+5\alpha)}{-120\alpha^3-216\alpha^5+420+30\alpha^2+220\alpha^4+105\alpha^6+40\alpha^7+\alpha^{10}-30\alpha^8}+$$
$$+\frac{10(x+\alpha)^3(\alpha^6-6\alpha^5+40\alpha^3+15\alpha^2-126\alpha-14)}{-120\alpha^3-216\alpha^5+420+30\alpha^2+220\alpha^4+105\alpha^6+40\alpha^7+\alpha^{10}-30\alpha^8}-$$
$$-\frac{15(x+\alpha)^4(20\alpha^2-21-5\alpha^4+\alpha^5+5\alpha)}{-120\alpha^3-216\alpha^5+420+30\alpha^2+220\alpha^4+105\alpha^6+40\alpha^7+\alpha^{10}-30\alpha^8}-$$
$$-\frac{(x+\alpha)^5(\alpha^6-6\alpha^5+40\alpha^3+15\alpha^2-126\alpha-14)}{-120\alpha^3-216\alpha^5+420+30\alpha^2+220\alpha^4+105\alpha^6+40\alpha^7+\alpha^{10}-30\alpha^8}+$$
$$+\frac{(x+\alpha)^6(20\alpha^2-21-5\alpha^4+\alpha^5+5\alpha)}{-120\alpha^3-216\alpha^5+420+30\alpha^2+220\alpha^4+105\alpha^6+40\alpha^7+\alpha^{10}-30\alpha^8} \quad (32)$$

Substitute any value of α into the interval [0,1], we obtain the approximate solution of the α-PDTM, Let $\alpha=1$. Now that the DO algorithm has been used to optimize the parameter α, better results are obtained by substituting the value of $\alpha^{DO}=0.22132$ which will be substituted into (32) to produce the approximate solution of α^{DO}-PDTM, as shown in Table 2 and Figure 1.

Table 2. Comparison of MAE and MSE for α-PDTM and α^{DO}-PDTM

Error Criteria	α-PDTM	αDO-PDTM
MSE	3.7430×10^{-2}	2.2298×10^{-5}
MAE	2.7889×10^{-1}	6.4054×10^{-3}

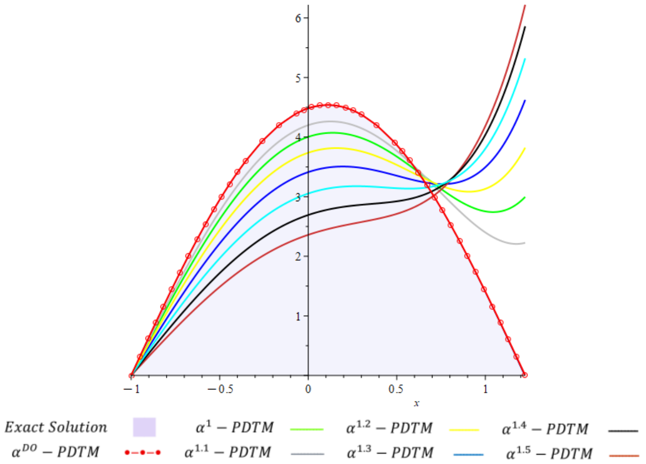

Fig. 1. Illustrates the matching process between α-PDTM, α^{DO}-PDTM, and Exact Solution

4.2. Example 2

Let us consider the following homogeneous differential equation [7]:

$$y''''(x)+5y''(x)+4y(x)=0, x\in[-1,0] \quad (33)$$

with the initial conditions

$$y(0)=1, y'(0)=0, y''(0)=0, y'''(0)=1 \quad (34)$$

and

$$y_{exact}(x)=\frac{4}{3}cos(x)+\frac{1}{3}sin(x)-\frac{1}{3}cos(2x)-\frac{1}{6}sin(2x) \quad (35)$$

If α-PDTM is applied to both sides of (33) we obtain

$$D(y''''(x)+5y''(x)+4y(x),\alpha;k)=$$
$$=(k+1)(k+2)(k+3)(k+4)D(y,\alpha;k+4)+$$
$$+5(k+1)(k+2)D(y,\alpha;k+2)+4D(y,\alpha;k)=0 \quad (36)$$

from the definition of α-PDTM

$$y_\alpha(x)=\sum_{k=0}^{n} D(y,\alpha;k)(x-x_\alpha)^k \quad (37)$$

where n is the number of iterations, $x_\alpha=a\alpha+(1-b)\alpha$. We will denote that $D(y,\alpha;0)=A$, $D(y,\alpha;1)=B$, $D(y,\alpha;2)=C$, $D(y,\alpha;3)=D$, then we find the other values of $D(y,\alpha;k)$ from (36)

$$D(y,\alpha;4)=-\frac{5C}{12}-\frac{A}{6}, D(y,\alpha;5)=$$
$$-\frac{D}{4}-\frac{B}{30}, D(y,\alpha;6)=\frac{7C}{120}+\frac{A}{36}, D(y,\alpha;7)=$$
$$=\frac{D}{40}+\frac{B}{252}, D(y,\alpha;8)=-\frac{17C}{4032}-\frac{A}{480}, \ldots$$

Then we substitute the all values of $D(y,\alpha;k)$ into (37) to get the approximate solution

$$y(x,\alpha)=\sum_{k=0}^{6} D(y,\alpha;k)(x\alpha)^k=$$
$$=A+(x+\alpha)B-(x+\alpha)^2A-(x+\alpha)^3\frac{B}{3}+(x+\alpha)^4\frac{A}{6}+$$
$$+(x+\alpha)^5\frac{B}{30}-(x+\alpha)^6\frac{A}{90} \quad (38)$$

where $x_\alpha=-\alpha$ and. When we substitute the initial conditions into (38), we get:

$$y(0,\alpha)=A+B\alpha+C\alpha^2+D\alpha^3+\left(-\frac{5C}{12}-\frac{A}{6}\right)\alpha^4+$$
$$+\left(-\frac{D}{4}-\frac{B}{30}\right)\alpha^5+\left(\frac{7C}{120}+\frac{A}{36}\right)\alpha^6+\left(\frac{D}{40}+\frac{B}{252}\right)\alpha^7+$$
$$+\left(-\frac{17C}{4032}-\frac{A}{480}\right)\alpha^8=1 \quad (39)$$
$$y'(0,\alpha)=A+B\alpha+C\alpha^2+D\alpha^3+\left(-\frac{5C}{12}-\frac{A}{6}\right)\alpha^4+$$
$$+\left(-\frac{D}{4}-\frac{B}{30}\right)\alpha^5+\left(\frac{7C}{120}+\frac{A}{36}\right)\alpha^6+\left(\frac{D}{40}+\frac{B}{252}\right)\alpha^7+$$
$$+\left(-\frac{17C}{4032}-\frac{A}{480}\right)\alpha^8=0 \quad (40)$$

$$y''(0,\alpha) = 2C + 6D\alpha + 12\left(-\frac{5C}{12} - \frac{A}{6}\right)\alpha^2 +$$
$$+20\left(-\frac{D}{4} - \frac{B}{30}\right)\alpha^3 + 30\left(\frac{7C}{120} + \frac{A}{36}\right)\alpha^4 + 42\left(\frac{D}{40} + \frac{B}{252}\right)\alpha^5 +$$
$$+56\left(-\frac{17C}{4032} - \frac{A}{480}\right)\alpha^6 = 0 \quad (41)$$

$$y'''(0,\alpha) = 6D + 24\left(-\frac{5C}{12} - \frac{A}{6}\right)\alpha + 60\left(-\frac{D}{4} - \frac{B}{30}\right)\alpha^2 +$$
$$+120\left(\frac{7C}{120} + \frac{A}{36}\right)\alpha^3 + 210\left(\frac{D}{40} + \frac{B}{252}\right)\alpha^4 +$$
$$+336\left(-\frac{17C}{4032} - \frac{A}{480}\right)\alpha^5 = 1 \quad (42)$$

When solving the (39), (40), (41) and (42), we will obtain the values of A, B, C and D

$A = -15(9639000\alpha^9 - 222264000\alpha^6 + 510\alpha^{17} + 6350400\alpha^7 - 40148\alpha^{15} + 6646080\alpha^{12} - 142800\alpha^{14} + 795375\alpha^{13} + 254016000\alpha^3 - 5255460\alpha^{11} + 180457200\alpha^8 - 58350600\alpha^{10} - 63504000\alpha^5 + 254016000\alpha^4 - 1524096000)/\big(4(-100\alpha^{18} + 4\alpha^{20} - 664807500\alpha^8 + 674730000\alpha^6 + 47250\alpha^{14} + 1080\alpha^{16} + 2685375\alpha^{12} + 5715360000 + 75836250\alpha^{10})\big)$

$B = -105(2\alpha^{16} - 390\alpha^{14} - 672\alpha^{13} + 14955\alpha^{12} + 86400\alpha^{11} - 161100\alpha^{10} - 1711440\alpha^9 + 680040\alpha^8 + 7956000\alpha^7 - 1377000\alpha^6 - 1814400\alpha^5 - 3175200\alpha^4 + 18144000\alpha^3 + 22680000\alpha^2 - 72576000\alpha - 54432000)\alpha^2/\big(2(-100\alpha^{18} + 4\alpha^{20} - 664807500\alpha^8 + 674730000\alpha^6 + 47250\alpha^{14} + 1080\alpha^{16} + 2685375\alpha^{12} + 5715360000 + 75836250\alpha^{10})\big)$

$C = (105(18\alpha^{16} - 1420\alpha^{14} - 5040\alpha^{13} + 28185\alpha^{12} + 235200\alpha^{11} - 186300\alpha^{10} - 2068920\alpha^9 + 340200\alpha^8 + 6426000\alpha^7 - 6350400\alpha^5 - 9525600\alpha^4 + 45360000\alpha^3 + 45360000\alpha^2 - 108864000\alpha - 54432000)\alpha)/(2(-100\alpha^{18} + 4\alpha^{20} - 664807500\alpha^8 + 674730000\alpha^6 + 47250\alpha^{14} + 1080\alpha^{16} + 2685375\alpha^{12} + 5715360000 + 75836250\alpha^{10}))$

$D = 5\big(10\alpha^{18} - 1998\alpha^{16} - 3360\alpha^{15} + 77175\alpha^{14} + 443520\alpha^{13} - 832860\alpha^{12} - 8845200\alpha^{11} + 3515400\alpha^{10} + 41277600\alpha^9 - 8334900\alpha^8 - 67473000\alpha^6 + 400075200\alpha^5 + 500094000\alpha^4 - 1905120000\alpha^3 - 1428840000\alpha^2 + 2286144000\alpha + 571536000)/\big(3(-100\alpha^{18} + 4\alpha^{20} - 664807500\alpha^8 + 674730000\alpha^6 + 47250\alpha^{14} + 1080\alpha^{16} + 2685375\alpha^{12} + 5715360000 + 75836250\alpha^{10})\big)$

When we substitute the values of A, B, C and D into (38). When we substitute any value of α into the interval [0,1], we obtain the approximate solution of the α-PDTM, Let $\alpha = 1$. Now that the DO algorithm has been used to optimize the parameter α, better results are obtained by substituting the value of $\alpha^{DO} = 0.71536$ which will be substituted into (38) to produce the approximate solution of α^{DO}-PDTM, as shown in Table 3 and Figure 2.

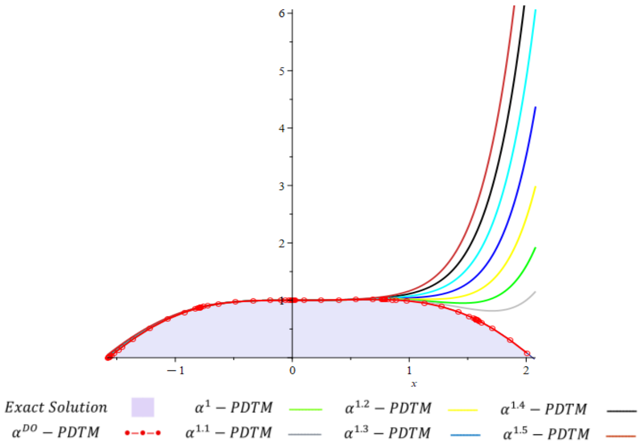

Fig. 2. Illustrates the matching process between α-PDTM, α^{DO}-PDTM, and Exact Solution

Table 3. Comparison of MAE and MSE for α-PDTM and α^{DO}-PDTM

Error Criteria	α-PDTM	αDO-PDTM
MSE	7.4254×10^{-3}	2.3236×10^{-5}
MAE	2.0443×10^{-2}	3.5810×10^{-3}

4.3. Example 3

Let us consider the following differential equation [3]:

$$y^{(6)}(x) - y(x) = -6e^x, x \in [0,1] \quad (43)$$

with the conditions

$$y(0) = 1,\ y'(0) = 0, y''(0) = -1$$
$$y(1) = 0, y'(1) = -e, u''(1) = -2e \quad (44)$$

and the exact solution when

$$y(x) = (1-x)e^x \quad (45)$$

If α-PDTM is applied to both sides of (43) we obtain

$$D\big(y^{(6)}(x) - y(x) + 6e^x, \alpha; k\big) =$$
$$(k+1)(k+2)(k+3)(k+4)(k+5)(k+6)D(y,\alpha;k+6) +$$
$$-D(y,\alpha;k) + \left(\frac{1^k}{k!}\right) = 0 \quad (46)$$

from the definition of α-PDTM

$$y_\alpha(x) = \sum_{k=0}^{n} D(y,\alpha;k)(x - x_\alpha)^k \quad (47)$$

where n is the number of iterations, $x_\alpha = a\alpha + (1-b)\alpha$. We will denote that $D(y,\alpha;0) = A$, $D(y,\alpha;1) = B$, $D(y,\alpha;2) = C$, $D(y,\alpha;3) = W$, $D(y,\alpha;4) = E$ and $D(y,\alpha;5) = f$, then we find the other values of $D(y,\alpha;k)$ from (46) $D(y,\alpha;2) = -A$, $D(y,\alpha;3) = \frac{-B}{3}$, $D(y,\alpha;4) = \frac{A}{6}$, $D(y,\alpha;5) = \frac{B}{30}$, $D(y,\alpha;6) = \frac{-A}{90}, \ldots$

Then we substitute the previous values of $D(y,\alpha;k)$ into (46) to get the approximate solution

$$y(x,\alpha) = \sum_{k=0}^{6} D(y,\alpha;k)(x+\alpha)^k =$$
$$= A + (x+\alpha)B - (x+\alpha)^2 A - (x+\alpha)^3\frac{B}{3} + (x+\alpha)^4\frac{A}{6} +$$
$$+(x+\alpha)^5\frac{B}{30} - (x+\alpha)^6\frac{A}{90} \quad (48)$$

where $x_\alpha = 1 - \alpha$, substitute the conditions of (44) into (48) to get:

$$y(0,\alpha) = A + \left(\frac{B}{2} - 3\right)(-1+\alpha)^7 + \left(\frac{C}{3} - 1\right)(-1+\alpha)^8 +$$
$$+\left(\frac{W}{4} - \frac{1}{4}\right)(-1+\alpha)^9 + \left(\frac{E}{5} - \frac{1}{20}\right)(-1+\alpha)^{10} +$$
$$+\left(\frac{F}{6} - \frac{1}{120}\right)(-1+\alpha)^{11} + \left(\frac{A}{7} - \frac{103}{120}\right)(-1+\alpha)^{12} = 1 \quad (49)$$

$$y'(0,\alpha) = B + 2C(-1+\alpha) + 3W(-1+\alpha)^2 + 4E(-1+\alpha)^3 +$$
$$+5F(-1+\alpha)^4 + 6(A-6)(-1+\alpha)^5 + 7\left(\frac{B}{2} - 3\right)(-1+\alpha)^6 +$$
$$+8\left(\frac{C}{3} - 1\right)(-1+\alpha)^7 + 9\left(\frac{W}{4} - \frac{1}{4}\right)(-1+\alpha)^8 +$$
$$+10\left(\frac{E}{5} - \frac{1}{20}\right)(-1+\alpha)^9 + 11\left(\frac{F}{6} - \frac{1}{120}\right)(-1+\alpha)^{10} +$$
$$+12\left(\frac{A}{7} - \frac{103}{120}\right)(-1+\alpha)^{11} = 0 \quad (50)$$

$$y''(0,\alpha) = 2C + 6W(-1+\alpha) + 12E(-1+\alpha)^2 +$$
$$+20F(-1+\alpha)^3 + 30(A-6)(-1+\alpha)^4 +$$
$$+42\left(\frac{B}{2} - 3\right)(-1+\alpha)^5 + 56\left(\frac{C}{3} - 1\right)(-1+\alpha)^6 +$$
$$+72\left(\frac{W}{4} - \frac{1}{4}\right)(-1+\alpha)^7 + 90\left(\frac{E}{5} - \frac{1}{20}\right)(-1+\alpha)^8 +$$
$$+110\left(\frac{F}{6} - \frac{1}{120}\right)(-1+\alpha)^9 + 132\left(\frac{A}{7} - \frac{103}{120}\right)(-1+\alpha)^{10} = -1 \quad (51)$$

$$y(1,\alpha) = A + B\alpha + C\alpha^2 + W\alpha^3 + E\alpha^4 + F\alpha^5 + (A-6)\alpha^6 +$$
$$+\left(\frac{B}{2} - 3\right)\alpha^7 + \left(\frac{C}{3} - 1\right)\alpha^8 + \left(\frac{W}{4} - \frac{1}{4}\right)\alpha^9 + \left(\frac{E}{5} - \frac{1}{20}\right)\alpha^{10} +$$
$$+\left(\frac{F}{6} - \frac{1}{120}\right)\alpha^{11} + \left(\frac{A}{7} - \frac{103}{120}\right)\alpha^{12} = 0 \quad (52)$$

$$y'^{(1,\alpha)} = B + 2C\alpha + 3W\alpha^2 + 4E\alpha^3 + 5F\alpha^4 + 6(A-6)\alpha^5 + +7\left(\frac{B}{2}-3\right)\alpha^6 + 8\left(\frac{C}{3}-1\right)\alpha^7 + 9\left(\frac{W}{4}-\frac{1}{4}\right)\alpha^8 + +10\left(\frac{E}{5}-\frac{1}{20}\right)\alpha^9 + 11\left(\frac{F}{6}-\frac{1}{120}\right)\alpha^{10} + +12\left(\frac{A}{7}-\frac{103}{120}\right)\alpha^{11} = -e \quad (53)$$

$$y(1,\alpha) = 2C + 6W\alpha + 12E\alpha^2 + 20F\alpha^3 + 30(A-6)\alpha^4 + +42\left(\frac{B}{2}-3\right)\alpha^5 + 56\left(\frac{C}{3}-1\right)\alpha^6 + 72\left(\frac{W}{4}-\frac{1}{4}\right)\alpha^7 + +90\left(\frac{E}{5}-\frac{1}{20}\right)\alpha^8 + 110\left(\frac{F}{6}-\frac{1}{120}\right)\alpha^9 + +132\left(\frac{A}{7}-\frac{103}{120}\right)\alpha^{10} = -2e \quad (54)$$

When solving (49)–(54), we obtain the values of A, B, C, W, E, and F, which we will substitute in (46) to obtain the final approximate solution of the α-PDTM.

$$y(x,\alpha) = A + B(x-1+\alpha) + C(x-1+\alpha)^2 + +W(x-1+\alpha)^3 + E(x-1+\alpha)^4 + F(x-1+\alpha)^5 + +(A-6)(x-1+\alpha)^6 + \left(\frac{B}{2}-3\right)(x-1+\alpha)^7 + +\left(\frac{C}{3}-1\right)(x-1+\alpha)^8 + \left(\frac{W}{4}-\frac{1}{4}\right)(x-1+\alpha)^9 + +\left(\frac{E}{5}-\frac{1}{20}\right)(x-1+\alpha)^{10} + \left(\frac{F}{6}-\frac{1}{120}\right)(x-1+\alpha)^{11} + +\left(\frac{A}{7}-\frac{103}{120}\right)(x-1+\alpha)^{12} \quad (55)$$

When we substitute any value of α between 0 and 1, we obtain the approximate solution of the α-PDTM, let $\alpha = 0.5$. Now that the DO algorithm has been used to optimize the parameter α, better results are obtained by substituting the value of $\alpha^{DO}=1$ which will be substituted into (55) to produce the approximate solution of α^{DO}-PDTM, as shown in Table 4 and Figure 3.

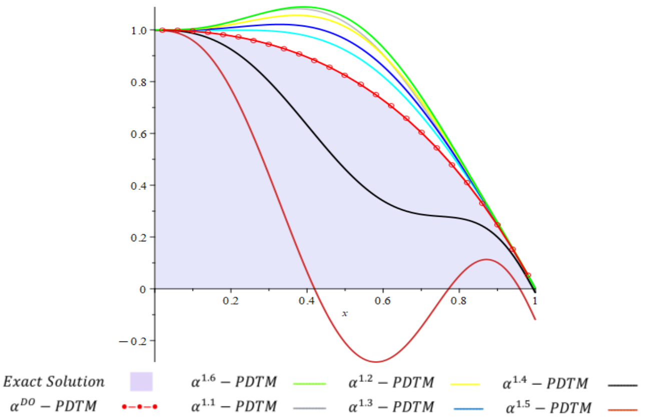

Fig. 3. Illustrates the matching process between α-PDTM, α^{DO}-PDTM, and Exact Solution

Table 4. Comparison of MAE and MSE for α-PDTM and α^{DO}-PDTM

Error Criteria	α-PDTM	αDO-PDTM
MSE	2.9794×10^{-3}	9.6221×10^{-4}
MAE	9.7371×10^{-2}	6.1858×10^{-2}

5. Conclusion

In this paper, a hybrid method between α-PDTM and DO algorithm is applied. A series of approximate solutions were applied in α-PDTM as a fitness function in the DO algorithm to find the best value for the parameter α. The results of the α-PDTM (which contains a random value for the parameter α) were compared with α^{DO}-PDTM (which contains the parameter α chosen by the DO algorithm) through three examples shown in Tables 2–4 and Figures 1–3, where α^{DO}-PDTM shows superior results on α-PDTM by calculating MSE and MAE.

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