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PERIODIC ATEB-FUNCTIONS AND THE VAN DER POL METHOD FOR CONSTRUCTING SOLUTIONS OF TWO-DIMENSIONAL NONLINEAR OSCILLATIONS MODELS OF ELASTIC BODIES

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Abstract. *In the process of operation, the simplest elements (hereinafter elastic bodies) of machines and mechanisms under the influence of external and internal factors carry out complex oscillations – a combination of longitudinal, bending and torsion combinations in various combinations. In general, mathematical models of the process of such complex phenomena in elastic bodies, even for one-dimensional calculation models, are boundary value problems for systems of partial differential equations. A two-dimensional mathematical model of oscillatory processes in a nonlinear elastic body is considered. A method of constructing an analytical solution of the corresponding boundary-value problems for nonlinear partial differential equations is proposed, which is based on the use of Ateba functions, the van der Pol method, ideas of asymptotic integration, and the principle of single-frequency oscillations. For "undisturbed" analogues of the model equations, single-frequency solutions were obtained in an explicit form, and for "perturbed" – analytical dependences of the basic parameters of the oscillation process on a small perturbation. The dependence of the main frequency of oscillations on the amplitude and non-linearity parameter of elastic properties in the case of single-frequency oscillations of "unperturbed motion" is established. An asymptotic approximation of the solution of the autonomous "perturbed" problem is constructed. Graphs of changes in amplitude and frequency of oscillations depending on the values of the system parameters are given.*

Keywords: oscillations, nonlinear elastic bodies, two-dimensional mathematical model

OKRESOWE FUNKCJE ATEB I METODA VAN DER POLA DO KONSTRUOWANIA ROZWIĄZAŃ DWUWYMIAROWYCH NIELINIOWYCH MODELI OSCYLACJI CIAŁ SPRĘŻYSTYCH

Streszczenie. *W procesie eksploatacji, najprostsze elementy (zwane dalej ciałami sprężystymi) maszyn i mechanizmów pod wpływem czynników zewnętrznych i wewnętrznych wykonują złożone oscylacje – wzdłużne, zginające i skręcające w różnych kombinacjach. Ogólnie rzecz biorąc, modele matematyczne procesu takich złożonych zjawisk w ciałach sprężystych, nawet dla jednowymiarowych modeli obliczeniowych, są problemami wartości brzegowych dla układów równań różniczkowych cząstkowych. Rozważany jest dwuwymiarowy model matematyczny procesów oscylacyjnych w nieliniowym ciele sprężystym. Zaproponowano metodę konstruowania analitycznego rozwiązania odpowiednich problemów wartości brzegowych dla nieliniowych równań różniczkowych cząstkowych, która opiera się na wykorzystaniu funkcji Ateba, metody van der Pola, idei całkowania asymptotycznego oraz zasady oscylacji jednoczęstotliwościowych. Dla „niezaburzonych" analogów równań modelu uzyskano rozwiązania jednoczęstotliwościowe w postaci jawnej, a dla „zaburzonych" – analityczne zależności podstawowych parametrów procesu oscylacji od niewielkiej perturbacji. Ustalono zależność głównej częstotliwości oscylacji od amplitudy i parametru nieliniowości właściwości sprężystych w przypadku jednoczęstotliwościowych oscylacji „ruchu niezaburzonego". Skonstruowano asymptotyczne przybliżenie rozwiązania autonomicznego problemu „zaburzonego". Podano wykresy zmian amplitudy i częstotliwości oscylacji w zależności od wartości parametrów układu.*

Słowa kluczowe: oscylacje, nieliniowe ciała sprężyste, dwuwymiarowy model matematyczny

Introduction

Analytical methods for the study of dynamic processes in elastic and flexible bodies (longitudinal, bending, torsional oscillations) are thoroughly developed on the basis of one-dimensional mathematical models for the case of linear, quasi-linear and, in some cases, nonlinear elastic properties of the material under various types of external actions: periodic or impulse. To describe the nonlinear processes of such systems and small motion disturbances, it is effective to use the combination of the principle of single-frequency oscillations with the basic ideas of asymptotic integration of boundary value problems for quasi-linear differential equations with partial derivatives or periodic Ateb functions. This applies to the case when the elastic properties of the material can be described by a nonlinear relationship $\sigma = E\varepsilon^{\nu+1} + \mu f(\varepsilon, \dot{\varepsilon})$, where σ, E, ε, respectively, the stress, "modulus of elasticity", the relative deformation of the material is an analytical function $\mu f(\varepsilon, \dot{\varepsilon})$ that describes a small deviation of the elastic properties of the material from the power law (this is indicated by a small parameter μ), $\nu+1 = \frac{2m+1}{2n+1}, m, n = 0, 1, 2, ...$ Oscillatory processes in such elastic bodies differ not only quantitatively, but also qualitatively, even in the case of quasi-linear ($\nu = 0$) elastic properties of an elastic body: the period of the dynamic process depends on the amplitude, and therefore a number of features of the actions of external periodic disturbances on these elastic bodies. However, with the use of one-dimensional mathematical models, it is not always possible to estimate with a sufficient degree of accuracy the influence of the geometric dimensions of an elastic body on the dynamics of the process in it. We are talking, first of all, about the influence of the ratio between length and width on the main parameters of oscillations. Such problems require consideration of more complex mathematical models of the dynamics of elastic bodies. This problem is partially solved in the work for a two-dimensional model of an elastic body, provided that its elastic properties are described by the above ratio.

The purpose of the paper: to develop a method of analytical construction of the solution of nonlinear problems of two-dimensional models of elastic bodies and to study their oscillatory processes for the case of homogeneous boundary conditions.

1. Analysis of recent research and publications

Analytical methods of studying the oscillatory processes of nonlinear systems with concentrated masses and distributed parameters under the continuous [6–8, 10, 15] or impulse action [2, 11, 12, 18, 22] of external factors have found a relatively wide development for one-dimensional models of elastic bodies. They are especially effective for practical application in cases where the dynamics for undisturbed analogues of the corresponding systems and the oscillatory process can be described using Fourier series or special Ateb functions [16, 17, 19]. The application of the main ideas of the asymptotic integration of boundary value problems for differential equations with partial derivatives or the van der Pol method [15, 23, 25] in combination with the principle of single-frequency oscillations in nonlinear systems for disturbed motion made it possible to establish, in particular, a number of features of the passage of resonance for systems with homogeneous boundary values conditions or systems characterized by a constant speed of longitudinal movement [1–5, 13, 21]. For the latter, on the basis of the basic ideas of the wave theory of motion, it was possible to establish the influence of the speed of longitudinal motion on the main parameters of the specified type of systems. With regard to the research of processes in nonlinear systems under the influence of periodic instantaneous disturbances, we note that such problems were considered mainly for systems with concentrated masses, and, in some cases, for the simplest models of systems with distributed parameters [22, 24, 26]. They substantiated the application of the averaging method for systems disturbed by instantaneous impulses, and it was proposed to represent the instantaneous action of external forces in mathematical models of oscillations of elastic bodies using delta functions [9, 27]. The main properties of the latter and the system of eigenfunctions, which describe the forms of undisturbed motion oscillations, make it possible to get rid of irregularities in the right-hand parts of mathematical models based on linear variables. Regarding the more complex two-dimensional mathematical models of the dynamics of elastic bodies, it should be noted that such problems were considered in partially two-dimensional models of elastic bodies for their linear or quasi-linear elastic properties [20]. Therefore, the development of a method of analytical research of oscillatory processes of two-dimensional models of elastic bodies, even under regular disturbances, is an urgent task.

2. Solving method

It is known [11] that the mathematical model of longitudinal oscillations of a one-dimensional elastic or flexible body, the material of which under certain boundary conditions satisfies the nonlinear relationship indicated above, is a differential equation

$$u_{tt} - \alpha_1^2 \left(u_x\right)^{\nu} u_{xx} = \mu F_1\left(u, u_t, u_x, u_{xx}\right) \tag{1}$$

in which $u(x,t)$ the longitudinal movement of the body with the coordinate x at an arbitrary moment in time t, $F_1\left(u, u_t, u_x, u_{xx}\right)$ a function expressed through the dependence of the deviation of the elastic properties of the body material from the power law, α_1^2 – became known.

Its analogue for a two-dimensional model of an elastic body is the equation

$$\begin{aligned} u_{tt} - \alpha^2 \left(u_x\right)^{\nu} u_{xx} - \beta^2 \left(u_y\right)^{\nu} u_{yy} &= \\ = \mu F_1\left(u, u_t, u_x, u_{xx}, u_y, u_{yy}\right) \end{aligned} \tag{2}$$

$t \geq 0$, $0 \leq x \leq l$, $0 \leq y \leq b$, in which α, β, ν, l, b – constants, $F_1\left(u, u_t, u_x, u_{xx}, u_y, u_{yy}\right)$ – a known analytic function.

The task is to construct a solution to equation (2) in the domain D: $t \geq 0$, $0 \leq x \leq l$, $0 \leq y \leq b$.

It should be noted that the issue of the existence of periodic solutions in unperturbed equations of the form (1) was considered in the paper [14], where only their period was established for both independent variables.

To construct a solution to the problem formulated above, we will use the general ideas of constructing solutions of equations with a small "disturbance". According to them, it is first necessary to construct the solution of the corresponding unperturbed ($\mu = 0$) equation.

Single-frequency oscillations of undisturbed motion.

It is easy to make sure that the method of separation of variables can be used at $\mu = 0$ to construct the solution of equation (2). According to it, we will look for the function $u(t,x,y)$ in the form

$$u\left(t,x,y\right) = T\left(t\right)V\left(x,y\right), \tag{3}$$

where $T\left(t\right)$ and $V\left(x,y\right)$ – unknown periodic functions on t, x, y, respectively.

To find them, according to the method of separation of variables, we obtain nonlinear differential equations

$$\ddot{T}\left(t\right) + \lambda\left(T\left(t\right)\right)^{\nu+1} = 0 \tag{4}$$

$$\begin{aligned} &\alpha^2\left(V_x\left(x,y\right)\right)^{\nu} V_{xx}\left(x,y\right) + \\ &+\beta^2\left(V_y\left(x,y\right)\right)^{\nu} V_{yy}\left(x,y\right) + \lambda V\left(x,y\right) = 0 \end{aligned} \tag{5}$$

where λ is an unknown parameter, the conditions for which will be considered below.

It is easy to verify that the linearly independent solutions of equation (5) are expressed in terms of periodic Ateb-functions in the form

$$V\left(x,y\right) = \mathrm{X}_0 \left\{ \begin{matrix} sa\left(1, \dfrac{1}{\nu+1}, \chi\left(\lambda, x, y\right)\right) \\ ca\left(1, \dfrac{1}{\nu+1}, \kappa\left(\lambda, x, y\right)\right) \end{matrix} \right\} \tag{6}$$

where X_0 is constant, and the appearance of the function $V\left(x,y\right)$, its arguments $\chi\left(\lambda,x,y\right)$ or $\kappa\left(\lambda,x,y\right)$ depends on the values of the function at the boundary of the area of change of independent linear variables. In particular, when a whole number of half-waves is "placed" in the specified area, the function and parameter take the corresponding values

$$\frac{V(x,y)}{\mathrm{X}_0} = \left\{ \begin{matrix} sa\left[1, \dfrac{1}{\nu+1}, \left(\lambda \dfrac{\nu+2}{2\mathrm{X}_0^{\nu}} \left(\begin{matrix} \alpha^2\left(\dfrac{m\tilde{\Pi}}{l}\right)^{\nu+2} + \\ +\beta^2\left(\dfrac{n\tilde{\Pi}}{b}\right)^{\nu+2} \end{matrix} \right)^{-1} \right)^{\frac{1}{\nu+2}} \left(\dfrac{m\tilde{\Pi}}{l}x + \dfrac{n\tilde{\Pi}}{b}y\right) \right] \\ ca\left[1,, \dfrac{1}{\nu+1}, \left(\lambda \dfrac{\nu+2}{2\mathrm{X}_0^{\nu}} \left(\begin{matrix} \alpha^2\left(\dfrac{m\tilde{\Pi}}{l}\right)^{\nu+2} + \\ +\beta^2\left(\dfrac{n\tilde{\Pi}}{b}\right)^{\nu+2} \end{matrix} \right)^{-1} \right)^{\frac{1}{\nu+2}} \left(\dfrac{m\tilde{\Pi}}{l}x + \dfrac{n\tilde{\Pi}}{b}y\right) \right] \end{matrix} \right\} \tag{7}$$

$$\lambda = \left[\alpha^2\left(\frac{m\tilde{\Pi}}{l}\right)^{\nu+2} + \beta^2\left(\frac{n\tilde{\Pi}}{b}\right)^{\nu+2}\right]\frac{2\mathrm{X}_0^{\nu}}{\nu+2}, \quad m, n = 1, 2, \ldots \tag{8}$$

In the above expressions, the half-period of the Ateb-functions is used for the third argument, i.e. $\tilde{\Pi} = \Pi\left(1, \frac{1}{\nu+1}\right) = \sqrt{\pi}\Gamma\left(\frac{\nu+1}{\nu+2}\right)\left(\Gamma\left(\frac{1}{2}+\frac{\nu+1}{\nu+2}\right)\right)^{-1}$. The results obtained above allow us to simultaneously construct the solution of equation (4):

$$T(t) = T_0 \begin{Bmatrix} ca\left(\nu+1, 1, \left(\frac{\nu+2}{2}\lambda T_0^{\nu}\right)^{\frac{1}{2}} t\right) \\ sa\left(\nu+1, 1, \left(\frac{\nu+2}{2}\lambda T_0^{\nu}\right)^{\frac{1}{2}} t\right) \end{Bmatrix}$$

or taking into account (8), we have

$$u(t,x,y) = \\ = a\begin{Bmatrix} ca\left[\left(\nu+1, 1, \omega(a)t+\theta\right)\right] sa\left[1, \frac{1}{\nu+1}, \left(\frac{m\tilde{\Pi}}{l}x + \frac{n\tilde{\Pi}}{b}y\right)\right] \\ sa\left[\left(\nu+1, 1, \omega(a)t+\theta\right)\right] ca\left[1, \frac{1}{\nu+1}, \left(\frac{m\tilde{\Pi}}{l}x + \frac{n\tilde{\Pi}}{b}y\right)\right] \end{Bmatrix} \quad (9)$$

$$\omega(a) = \sqrt{\left[\alpha^2\left(\frac{m}{l}\right)^{\nu+2} + \beta^2\left(\frac{n}{b}\right)^{\nu+2}\right]\left(\tilde{\Pi}\right)^{\nu+2}} a^{\frac{\nu}{2}} \quad (10)$$

where $a = X_0 T_0$ is amplitude of a single-frequency process of undisturbed motion, $\omega(a)$ – frequency, θ – its initial phase.

Fig. 1 shows the dependence of the main frequency of oscillations on the amplitude and the nonlinearity parameter for the following characteristics of the system $\alpha^2 = 20$, $\beta^2 = 0.36\alpha^2$, $l = 1$, $b = 0.1l$ – a), b); $\alpha^2 = 20$, $\beta^2 = 0.36\alpha^2$, $l = 1$, $b = 0.2l$ – c), d).

It follows from the given graphic dependences that for the case when the elastic properties of the material of an elastic body are described by a nonlinear relationship $\sigma = E\varepsilon^{\nu+1}$ depend on the amplitude). Under condition $\nu > 0$, a larger value of the oscillation amplitude corresponds to a larger value of the natural frequency, and vice versa, with a larger value of the oscillation amplitude, the value of the natural frequency is smaller (in this case, the natural frequency does not depend on the amplitude).

The latter properties are especially relevant when studying the impact of periodic disturbance on the object under study. Such a task can be the subject of separate studies.

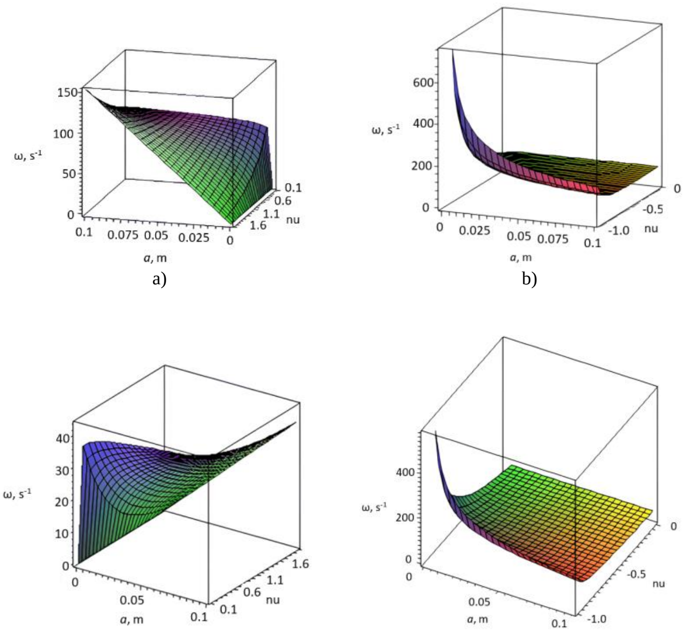

Fig. 1. The dependence of the natural frequency of oscillations of a two-dimensional body on the amplitude and the parameter that characterizes its elastic properties for different geometric dimensions.

3. Construction of the asymptotic approximation of the autonomous perturbed problem

As already emphasized above, the maximum value of the right-hand side of equation (2) is a small value in comparison with the maximum values of the terms of its left-hand side. Therefore, to construct an analytical solution to the problem (determining the influence of its right-hand side on an oscillating single-frequency process), you can use the general ideas of perturbation methods, more precisely, the asymptotic method of the KBM (Krylov-Bogolyubov-Mytropolskyi) or adapt the main ideas of the van der Pol method for it. Below we will present the main results that follow from the van der Pol method, which is simpler in terms of mathematical calculations and much more convenient for engineering calculations. According to his main idea, small forces cause for short systems a slow change in time only in the amplitude and frequency of oscillations [15]. Thus, we will look for the solution of equation (2) in the form of (10) with the only difference that in it α and θ are slowly varying functions of time. Below, for undisturbed motion, we will take the first of these ratios. Thus, for disturbed motion, the solution of the basic problem must be sought in the form

$$u(t,x,y) = a(t)\, ca(\nu+1, 1, \psi)\, sa\left(1, \frac{1}{\nu+1}, \Theta(x,y)\right) \quad (11)$$

where $\psi = \omega(a(t))t + \theta(t), \Theta(x,y) = \tilde{\Pi}\cdot\left(\frac{x}{b}+\frac{y}{c}\right)$.

The problem is to determine such functions of time $a(t)$ and $\theta(t)$, for which relation (11) satisfies equation (2). Therefore, after differentiating relation (11) with respect to time, we obtain

$$\frac{\partial u}{\partial t}=sa\left(1,\frac{1}{\nu+1},\Theta(x,y)\right)\left\{\begin{array}{l}ca(\nu+1,1,\psi)\frac{da}{dt}-\frac{2}{\nu+2}a\omega(a)sa(1,\nu+1,\psi)-\\-\frac{2}{\nu+2}a\omega(a)sa(1,\nu+1,\psi)\frac{d\theta}{dt}\end{array}\right\}$$

$$\frac{\partial^2 u}{\partial t^2}=-\frac{2}{\nu+2}\left[\begin{array}{l}\frac{da}{dt}\left(\omega(a)+a\frac{d\omega}{da}\right)sa(1,\nu+1,\psi)-\\-a\omega(a)\left(\omega(a)+\frac{d\vartheta}{dt}\right)ca^{\nu+1}(\nu+1,1,\psi)\end{array}\right]sa\left(1,\frac{1}{\nu+1},\Theta(x,y)\right) \tag{12}$$

$$\frac{\partial u}{\partial x}=a\frac{\tilde{\Pi}}{b}\left(ca\left(\frac{1}{\nu+1},1,\Theta(x,y)\right)\right)^{\frac{1}{\nu+1}}ca(\nu+1,1,\psi)$$

$$\frac{\partial^2 u}{\partial x^2}=-\frac{2}{\nu+2}a\left(\frac{\tilde{\Pi}}{b}\right)^2\left(ca\left(\frac{1}{\nu+1},1,\Theta(x,y)\right)\right)^{\frac{-\nu}{\nu+1}}sa\left(1,\frac{1}{\nu+1},\Theta(x,y)\right)ca(\nu+1,1,\psi) \tag{13}$$

$$\frac{\partial u}{\partial y}=a\frac{\tilde{\Pi}}{c}\left(ca\left(\frac{1}{\nu+1},1,\Theta(x,y)\right)\right)^{\frac{1}{\nu+1}}ca(\nu+1,1,\psi)$$

$$\frac{\partial^2 u}{\partial y^2}=-\frac{2}{\nu+2}a\left(\frac{\tilde{\Pi}}{c}\right)^2\left(ca\left(\frac{1}{\nu+1},1,\Theta(x,y)\right)\right)^{\frac{-\nu}{\nu+1}}sa\left(1,\frac{1}{\nu+1},\Theta(x,y)\right)ca(\nu+1,1,\psi) \tag{14}$$

It should be noted that in the expression for the second time derivative of the function in accordance with the general idea of the van der Pol method, adapted to systems with distributed parameters, the relation

$$sa\left(1,\frac{1}{\nu+1},\Theta(x,y)\right)\cdot$$
$$\cdot\left\{ca(\nu+1,1,\psi)\frac{da}{dt}-\frac{2}{\nu+2}a\omega(a)sa(1,\nu+1,,\psi)\frac{d\theta}{dt}\right\}=0 \tag{15}$$

If we substitute the above dependencies into the basic equation (2), we get

$$\frac{2}{\nu+2}\left[\frac{da}{dt}\left(\omega(a)+a\frac{d\omega}{da}\right)sa(1,\nu+1,\psi)+\right.$$
$$\left.+a\omega(a)\frac{d\theta}{dt}ca^{\nu+1}(\nu+1,1,\psi)\right]= \tag{16}$$
$$=-\mu\frac{1}{\mathrm{P}}\int_0^l\int_0^b F(a,x,y,\psi)sa\left(1,\frac{1}{\nu+1},\Theta(x,y)\right)dxdy$$

where $F(a,x,y,\psi)$ – corresponds to the value of the function $F_1(u,u_t,u_x,u_{xx},u_y,u_{yy})$ under the conditions $u(t,x,y)$ defined in it according to dependence (9) and $\mathrm{P}=\frac{1}{l}\frac{1}{b}\int_0^l\int_0^b sa^2\left(1,\frac{1}{\nu+1},\left(\frac{\tilde{\Pi}}{l}x+\frac{\tilde{\Pi}}{b}y\right)\right)dxdy.$

Dependencies (15), (16) determine the main parameters of oscillations of the studied system by ordinary differential equations

$$\dot{a}=-\mu\frac{sa(1,\nu+1,\psi)}{\omega(a)\mathrm{P}}\int_0^l\int_0^b F(a,x,y,\psi)sa\left(1,\frac{1}{\nu+1},\Theta(x,y)\right)dxdy$$

$$\dot{\psi}=\omega(a)-\frac{(\nu+2)ca^{\nu+1}(\nu+1,1,\psi)}{2a\omega(a)\mathrm{P}}\cdot \tag{17}$$
$$\cdot\int_0^l\int_0^b F(a,x,y,\psi)sa\left(1,\frac{1}{\nu+1},\Theta(x,y)\right)dxdy$$

If we take into account that during one period of oscillations, small-scale motion disturbances in non-linear systems of the autonomous type cause a small-scale change in the amplitude and period of oscillations, then after averaging the system of equations (17) by phase, we will obtain equations in the standard form for describing the main parameters of system oscillations

$$\dot{a}=\mu A(a)=-\frac{\mu}{2\omega(a)\Pi\mathrm{P}}\cdot$$
$$\cdot\int_0^{2\Pi}\int_0^l\int_0^b sa(\nu+1,1,\psi)F(a,x,y,\psi)sa\left(1,\frac{1}{\nu+1},\Theta(x,y)\right)dxdyd\psi$$
$$\dot{\psi}=\omega(a)+\mu B(a)=\omega(a)-\frac{(\nu+2)}{4a\omega(a)\Pi\mathrm{P}}\cdot \tag{18}$$
$$\cdot\int_0^{2\Pi}\int_0^l\int_0^b ca^{\nu+1}(1,\nu+1,\psi)F(a,x,y,\psi)\,sa\left(1,\frac{1}{\nu+1},\Theta(x,y)\right)dxdyd\psi$$

where 2Π – the period by ψ functions $sa(1,\nu+1,\psi)$ and $ca(\nu+1,1,\psi)$ i.e. $2\Pi=\sqrt{\pi}\Gamma\left(\frac{1}{\nu+2}\right)\Gamma^{-1}\left(\frac{1}{2}+\frac{1}{\nu+2}\right)$.

Below, in Fig. 2, in accordance with dependence (18), for the case $F_1(u,u_t,u_x,u_{xx},u_y,u_{yy})=-k_1u_t^{2s+1}$ changes in the amplitude and frequency of oscillations over time are shown.

The given graphical dependencies show that the main parameters of the system, which characterize nonlinear forces and the force of resistance, do not affect the qualitative picture of changes in the amplitude of oscillations over time. As for the frequency of oscillations, it can increase over time at $-1<\nu<0$ and decrease at parameter $\nu>0$. At the same time, the rate of increase (decrease) of the frequency depends not only on the parameters that characterize the resistance force, but also on the parameter ν – characteristics of the nonlinear elastic properties of the material: for larger values of the specified parameter, the rate of decrease (increase) of the natural frequency is greater.

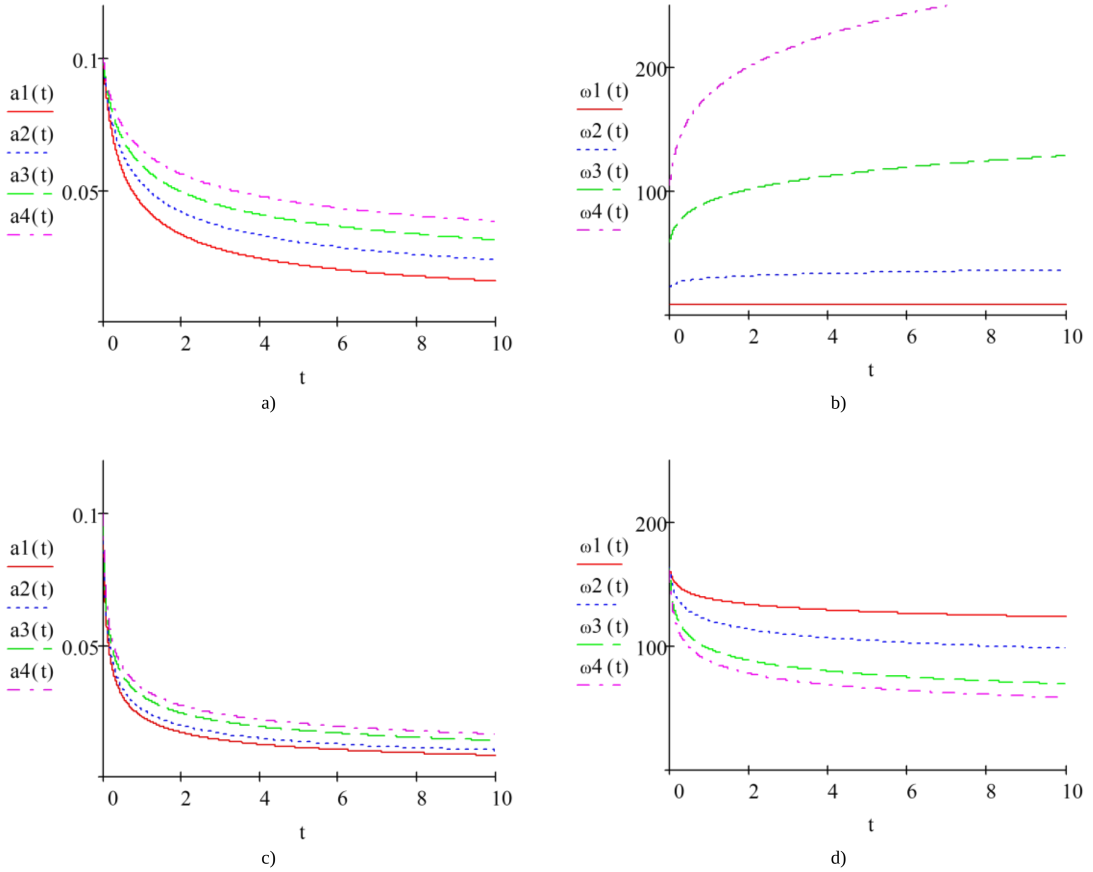

Fig. 2. Changes in time of amplitude a), c) and frequency b), d) of a single-frequency process for the following values of system parameters:

a), b): $\alpha^2 = 200$, $\beta^2 = 0.36\alpha^2$, $l = 1$, $b = 0.1l$, $k_1 = 3$, $s = 1$, $\nu = 0$ *red*, $\nu = \frac{-2}{7}$ *blue*, $\nu = \frac{-4}{7}$ *green*, $\nu = \frac{-6}{7}$ *orange*

c), d:) $\alpha^2 = 200$, $\beta^2 = 0.36\alpha^2$, $l = 1$, $b = 0.1l$, $k_1 = 3$, $s = 1$: $\nu = \frac{-2}{7}$ *red*, $\nu = \frac{4}{7}$ *blue*, $\nu = \frac{6}{7}$ *green*, $\nu = \frac{8}{7}$ *orange*

4. Conclusions

In this work, a technique for constructing approximate analytical solutions of nonlinear differential equations with partial derivatives was developed, which describe the vibration processes of two-dimensional "short" elastic bodies. The technique is based on the use of periodic Ateba functions to construct "unperturbed" analogues of the equations and to extend the general ideas of the van der Pol method to their perturbed analogues, provided that the latter are autonomous. For undisturbed analogues of the equations, single-frequency solutions are described in an explicit form, for perturbed ones, analytical dependences of the basic parameters of the process on a small perturbation are obtained. A feature of the single-frequency solutions of the corresponding equations is their dependence on the amplitude (initial conditions). It is shown that in the case of the progressive law of nonlinearity ($\nu > 0$), a larger value of the oscillation amplitude corresponds to a larger value of the natural frequency, and vice versa for the regressive law ($-1 < \nu < 0$): a larger value of the oscillation amplitude means a smaller value of the natural frequency (in the case of a linear law, the natural frequency does not depend on the amplitude).

It is also characteristic:

a) larger values of the geometric parameters of the elastic body (length and width) with all other constant parameters correspond to smaller values of the natural frequency;

b) the influence of the resistance force on the regularity of changes in time of the main parameters of the oscillations of the corresponding systems is manifested in the fact that the frequency increases over time for the regressive law of elastic properties and decreases for the progressive law of elastic properties.

The reliability of the obtained results is confirmed by obtaining in the limiting case ($\beta = 0$) known from literary sources.

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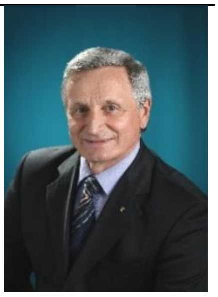

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In memory of B. I. Sokil as a scientist, teacher, friend and father