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OBJECTS FEATURES EXTRACTION BY SINGULAR PROJECTIONS OF DATA TENSOR TO MATRICES

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Abstract. *The problem of multidimensional tensor objects features extraction in a manner of matrices is considered. The tensor' elements Higher Order Singular Value Decomposition (SVD) is presented as the d-SVD which includes SVD of the tensor reshaped as a matrix and SVDs of reduced size of the previous SVDs vectors reshaped as matrices. The decomposition allows to create Singular Projections of tensor to a sum of one-rank tensors in selected dimensions. The projections of tensor to matrices by weighted and direct averaging in SVD' vectors subspace is investigated numerically. The extracted by projection features of a series of image objects are used to develop the optimized Inverse Feature Filters for the objects recognition.*

Keywords: higher order singular value decomposition, singular projection, objects recognition, optimized inverse features filters

WYODRĘBNIANIE CECH OBIEKTÓW POPRZEZ POJEDYNCZE RZUTOWANIA TENSORA DANYCH NA MACIERZE

Streszczenie. *Rozważano problem ekstrakcji cech wielowymiarowych obiektów tensorowych w postaci macierzy. Elementy rozkład wartości pojedynczej tensora wyższego rzędu przedstawiono jako d-SVD, który obejmuje rozkład na wartości własne (SVD) tensora przekształconego w macierz oraz rozkłady na wartości własne o zmniejszonym rozmiarze poprzednich wektorów własnych przekształconych w macierze. Dekompozycja pozwala na utworzenie projekcji osobliwych tensora na sumę tensorów pierwszego rzędu w wybranych wymiarach. Rzutowanie tensora na macierze poprzez ważone i bezpośrednie uśrednianie w podprzestrzeni wektorów własnych są badane numerycznie. Cechy wyodrębnione przez projekcję serii obiektów obrazowych są wykorzystywane do opracowania zoptymalizowanych odwrotnych filtrów cech do rozpoznawania obiektów.*

Słowa kluczowe: rozkład wartości osobliwych wyższego rzędu, projekcja osobliwa, rozpoznawanie obiektów, zoptymalizowane filtry cech odwrotnych

Introduction

Multidimensional data in signal processing, image objects recognition and economics' big data are presenting as mutually related series of matrices or vectors and it is commonly to call these data as tensors [2]. The data processing is making with an aim to find features of interested objects to recognize or analysis them. The main method to evaluate the data' characteristics is averaging of data series. The averaging blurs the characteristics and hide main features.

The tensor approach to representing multidimensional data is used to decompose data into components associated with different factors of influence on target problems. Also, a goal of the decompositions is to reduce the size of the tensor without losing significant data.

The Higher Order Singular Value Decomposition (HOSVD) [1, 2, 5] is using to separate tensor on independent one-rank components. Also, it is using to factorize tensors on sequential multipliers with the aim to reduce computational complexity and to extract data features by the way of projection to HOSVD' subspace [3, 6–8].

The HOSVD of a tensor' elements is offered in the form which allows to separate independent Singular Projections (SP) of the tensor associated with significant singular values in selected dimensions

1. Related works

The data array is represented as a d-dimensional tensor of complex or real elements

$$[x_{i_0;i_1;\ldots;i_{d-1}}]_{i_k=0\ldots N_k-1;k=0\ldots d-1} \tag{1.1}$$

of size $N_0 \times N_1 \times \ldots \times N_{d-1}$. The indices of array elements of different dimensions are separated by the sign ";", the indices of dimensions in the lexicographic one-dimensional (unfolded [2]) representation are not separated, multiplication of entities of different dimensions is denote as "×", matrices with different indices are different.

Let's consider techniques of using the SVD for d-dimensional array [2, 4, 7]. The technique [2, 7] looks as the following. The elements of tensor (1.1) are reshaped as d matrices of elements $x_{i_n;i_0\ldots i_{n-1}i_{n+1}\ldots i_{d-1}}$ and are defined SVDs of the matrices, their elements are as the following ones.

$$x_{i_n;i_0\ldots i_{n-1}i_{n+1}\ldots i_{d-1}} = \sum_{k_n} u_{i_n;k_n} s_{k_n} v_{i_0\ldots i_{n-1}i_{n+1}\ldots i_{d-1};k_n} \tag{1.2}$$

where $n = 0,\ldots,d-1$, $k_n = 0,\ldots,N_n - 1$ or k_n does not exceed value of the rank of the matrix in n-th dimension, $u_{i_n;k_n}$ are elements of N_n left orthogonal unitary vectors of size N_n and $v_{i_0\ldots i_{n-1}i_{n+1}\ldots i_{d-1};k_n}$ are elements of $\prod_{k=0\ldots d-1:k\neq n} N_k$ right orthogonal unitary vectors of size N_n, s_{k_n} – the elements of diagonal matrices of Singular Values (SVs). Are defined elements of the core tensor in the vector space

$$c_{k_0;\ldots;k_{d-1}} = \sum_{i_0} \cdots \sum_{i_{d-1}} x_{i_0;i_1;\ldots;i_{d-1}} u_{i_0;k_0} \ldots u_{i_{d-1};k_{d-1}} \tag{1.3}$$

which allow to present the HOSVD as

$$x_{i_0;\ldots;i_{d-1}} = \sum_{k_0} \cdots \sum_{k_{d-1}} c_{k_0;\ldots;k_{d-1}} u_{i_0;k_0} \ldots u_{i_{d-1};k_{d-1}} \tag{1.4}$$

If unfolded data matrices of elements (1.2) are not of full rank then the HOSVD yields core tensor (1.3) of smaller size than initial tensor (1.1). The SVDs (1.2) can be complicated due to the large sizes of the formed data matrices of elements (1.2), despite that only their left matrices need to be calculated. The matrices for objects recognition are formed on the base of HOSVD (1.4) in [7] as projections in lower dimension space.

The technique [4] is as follows. Elements of tensor (1.1), reshaped as a matrix, are presented by the SVD as

$$x_{i_0;i_1\ldots i_{d-1}} = \sum_{k_0} u_{i_0;k_0} s_{k_0} v_{i_1\ldots i_{d-1};k_0} \tag{1.5}$$

The matrix of right vectors can be reshaped and represented as

$$v_{k_0 i_1;i_2\ldots i_{d-1}} = \sum_{j_0 k_1} u_{k_0 i_1;j_0 k_1} s_{j_0 k_1} v_{i_2\ldots i_{d-1};j_0 k_1} \tag{1.6}$$

where joined indices $j_0 k_1 = 0,\ldots,N_0 N_1 - 1$. The reshaped matrix of right vectors as $v_{j_0 k_1 i_2;i_3\ldots i_{d-1}}$ is represented by SVD like (1.6) too and so on. The joining of the SVDs gives the HOSVD of the tensor elements

$$x_{i_0;...;i_{d-1}} = \sum_{k_0} u_{i_0;k_0} s_{k_0} \sum_{j_0 k_1} u_{k_0 i_1;j_0 k_1} s_{j_0 k_1} \cdots \sum_{j_0 \dots j_{d-2} k_{d-1}} u_{j_0 j_1 \dots i_{d-2};k_{d-1}} s_{j_0 \dots j_{d-2} k_{d-1}} v_{i_{d-1};k_{d-1}} \quad (1.7)$$

Decomposition (1.7) is effective if instead sums by joined indices $j_0 ... j_{d-2} k_{d-1} = 0,..., N_0 ... N_{d-1} - 1$ to use sums in a range of low rank approximation of the corresponding matrices. In general, the decomposition (1.7) needs in $d-1$ SVDs of the total size $\prod_{k=0...d-1} N_k$ with a decrease due to the rank considered.

Tensor (1.1) may be structured so, that some of its dimensions characterize an inner feature of data and some of them characterize the features differences along coordinates. The aim is to estimate and extract matrices of features. These features can be obtained by projection data tensor to matrices.

It is better to calculate SVD of some number of matrices of small size than of one matrix of large size. Therefore, it is offered the version of the techniques [4, 7] which includes only one SVD of maximum size.

2. The *d*-SVD

Tensor (1.1) can be reshaped as 2-dimensional matrix of size $\prod_{k=0...d-2} N_k \times N_{d-1}$ with elements $x_{i_0 ... i_{d-2};i_{d-1}}$. The SVD of the matrix' elements is the following.

$$x_{i_0 ... i_{d-2};i_{d-1}} = \sum_{k_{d-1}} u_{i_0 ... i_{d-2};k_{d-1}} s_{k_{d-1}} v_{i_{d-1};k_{d-1}} \quad (2.1)$$

Each k_{d-1}-th vector with elements $u_{i_0 ... i_{d-2};k_{d-1}}$ can be reshaped as a matrix of elements $u_{i_0 ... i_{d-3};i_{d-2};k_{d-1}}$ and represented by the SVD as

$$u_{i_0 ... i_{d-3};i_{d-2};k_{d-1}} = \sum_{k_{d-2}} u_{i_0 ... i_{d-3};k_{d-2};k_{d-1}} s_{k_{d-2};k_{d-1}} \cdot v_{i_{d-2};k_{d-2};k_{d-1}} \quad (2.2)$$

The substitution of (2.2) into (2.1) yields that

$$x_{i_0 ... i_{d-3};i_{d-2};i_{d-1}} = \sum_{k_{d-1}} s_{k_{d-1}} v_{i_{d-1},k_{d-1}} \cdot \sum_{k_{d-2}} u_{i_0 ... i_{d-3};k_{d-2};k_{d-1}} . s_{k_{d-2};k_{d-1}} v_{i_{d-2};k_{d-2};k_{d-1}} \quad (2.3)$$

The $k_{d-2}; k_{d-1}$-th vectors with elements $u_{i_0 ... i_{d-3};k_{d-2};k_{d-1}}$ in (2.3) can be reshaped as matrices and represented with using SVDs. And so on, it will be got finally the following representation of the original tensor' elements:

$$x_{i_0;...;i_{d-1}} = \sum_{k_{d-1}} s_{k_{d-1}} v_{i_{d-1};k_{d-1}} \sum_{k_{d-2}} s_{k_{d-2};k_{d-1}} v_{i_{d-2};k_{d-2};k_{d-1}} \cdots \sum_{k_1} u_{i_0;k_1;k_2;...;k_{d-1}} s_{k_1;k_2;...;k_{d-1}} v_{i_1;k_1;k_2;...;k_{d-1}} \quad (2.4)$$

As in usual SVD of 2-dimensional matrix, the number of sums levels in *d*-SVD (2.4) is equal to *d*-1, the multidimensional joined matrices of the singular values are block-diagonal.

The selection only main vectors with 0-th indices in (2.4) gives the one-rank component of the tensor' elements

$$x^{(0_1;...;0_{d-1})}_{i_0;...;i_{d-1}} = s_{0_{d-1}} v_{i_{d-1};0_{d-1}} s_{0_{d-2};0_{d-1}} v_{i_{d-2};0_{d-2};0_{d-1}} \cdots u_{i_0;0_1;...;0_{d-1}} s_{0_1;...;0_{d-1}} v_{i_1;0_1;...;0_{d-1}} \quad (2.5)$$

Similar components can be extracted for combinations of indices in brackets in (2.5) that correspond most significant SVs. These components are independent due to orthogonality of the vectors on each step of the *d*-SVD and therefore can be summarized to obtain an approximation of the initial tensor.

As it follows from (2.1)-(2.4), the tensor decomposition needs in one SVD of size $\prod_{k=0...d-2} N_k \times N_{d-1}$, N_{d-1} SVDs of size $N_0 N_1 ... N_{d-3} \times N_{d-2}$, $N_{d-2} N_{d-1}$ of size $N_0 N_1 ... N_{d-4} \times N_{d-3}$ and finally $N_2 ... N_{d-1}$ SVDs of size $N_0 \times N_1$. One-rank component (2.5) needs in one SVD of each size listed above.

The number of SVDs on each step of (2.4) can be determined by the number of significant SVs on the previous step. It is advisable to reshape tensor (1.1) as a matrix nearest to square one of size $N_0 ... N_{e-1} \times N_e ... N_{d-1}$ for selection of maximum number of significant SVs. The SVD of the matrix is the following.

$$x_{i_0 ... i_{e-1};i_e ... i_{d-1}} = \sum_{k=0}^{K-1} u_{i_0 ... i_{e-1};k} s_k v_{i_e ... i_{d-1};k} \quad (2.6)$$

where K is rank of the matrix. Each of K left and right vectors of SVD (2.6) may be reshaped as matrices like (2.2) in respect to dimensions, for example *e-1*-th and *d-1*-th, then the tensor

$$x_{i_0 ... i_{e-2};i_{e-1};i_e ... i_{d-1};i_{d-2}} = \sum_{k=0}^{K-1} s_k \sum_{k_{e-1}} u_{i_0 ... i_{e-2};k_{e-1};k} \cdot s_{k_{e-1};k} v_{i_{e-1};k_{e-1}k} \sum_{k_{d-1}} u_{i_e ... i_{d-2};k_{d-1};k} s_{k_{d-1};k} v_{i_{d-1};k_{d-1}k} \quad (2.7)$$

Decomposition (2.7) is the decomposition of the left and right vectors of SVD (2.6) separately to the form of *d*-SVD (2.4).

3. Tensor to matrices projections

As it follows from expressions (2.1) – (2.4), (2.7) the SVs of the first stage (2.1) and (2.6) directly relate to the tensor data. The SVs of the following stages like (2.2) are related to the structure of the vector space. Therefore, it can be defined the inverse tensor in respect to $d-1$-th dimension as

$$x^{-i_{d-1}}_{i_0;...;i_{d-2}} = \sum_{k_{d-1}} s^{-1}_{k_{d-1}} v_{i_{d-1};k_{d-1}} \sum_{k_{d-2}} s_{k_{d-2};k_{d-1}} v_{i_{d-2};k_{d-2};k_{d-1}} \cdots \sum_{k_1} u_{i_0;k_1;k_2;...;k_{d-1}} s_{k_1;k_2;...;k_{d-1}} v_{i_1;k_1;k_2;...;k_{d-1}} \quad (3.1)$$

This means that

$$\sum_{i_0;...;i_{d-2}} x^{-i_{d-1}}_{i_0;...;i_{d-2}} x_{i_0;...;i_{d-2};j_{d-1}} = \delta_{i_{d-1};j_{d-1}}$$

(δ – delta function) because of the orthogonality of the vectors of the SVD. The transform (3.1) can be used partially in relation to selected dimensions.

3.1. Projections by dimension reducing

The multiplication of tensor (2.4) by the matrix with elements $s^{-1}_{q_{d-1}} v_{i_{d-1};q_{d-1}}$ reduces the tensor' dimension, this looks as

$$\sum_{k_{d-1}} s_{k_{d-1}} s^{-1}_{q_{d-1}} \sum_{i_{d-1}} v_{i_{d-1};k_{d-1}} v_{i_{d-1};q_{d-1}} \cdots \sum_{k_1} u_{i_0;k_1;k_2;...;k_{d-1}} s_{k_1;k_2;...;k_{d-1}} v_{i_1;k_1;k_2;...;k_{d-1}} = x_{i_0;...;i_{d-2}} \quad (3.2)$$

Tensor' (2.4) elements related to $d-2$-th dimension $s_{k_{d-2};k_{d-1}} v_{i_{d-2};k_{d-2};k_{d-1}}$ can be compiled as a row of blocks, where i_{d-2} and k_{d-2} are indices of the block' columns and rows, k_{d-1} is blocks index. The multiplication by column of blocks of elements $s^{-1}_{q_{d-2};k_{d-1}} v_{i_{d-2};q_{d-2};k_{d-1}}$ gives block wise identity matrix:

$$\sum_{i_{d-2}} s_{k_{d-2};k_{d-1}} v_{i_{d-2};k_{d-2};k_{d-1}} \cdot s^{-1}_{q_{d-2};k_{d-1}} v_{i_{d-2};q_{d-2};k_{d-1}} = \delta_{k_{d-2};q_{d-2};k_{d-1}} \quad (3.3)$$

The multiplication like (3.3) of tensor (2.4) gives the tensor of reduced dimension

$$\sum_{k_{d-1}} s_{k_{d-1}} v_{i_{d-1};k_{d-1}} \sum_{k_{d-2};k_{d-3}} s_{k_{d-3};k_{d-2};k_{d-1}} v_{i_{d-3};k_{d-3}k_{d-2};k_{d-1}} \cdots \sum_{k_1} u_{i_0;k_1;k_2;\ldots;k_{d-1}} s_{k_1;k_2;\ldots;k_{d-1}} v_{i_1;k_1;k_2;\ldots;k_{d-1}} = x_{i_0;\ldots;i_{d-3};i_{d-1}} \quad (3.4)$$

It can be given a matrix in chosen two dimensions by same transforms of the tensor presented using *d*-SVD (2.4), (2.7).

3.2. General projections

The projection of a *d*-tensor on a matrix in dimensions $p\times q: q>p$ in terms of the *d*-SVD (2.4) can be formulated as the problem to minimize the functional

$$P_\varsigma(i_p,i_q) = \underset{\varsigma}{argmin} \sum_{i_0;\ldots;i_{d-1}} (x_{i_0;\ldots;i_{d-1}} - \sum_{k_p;\ldots;k_q;\ldots k_{d-1}} \varsigma_{k_p;\ldots;k_q;\ldots;k_{d-1}} v_{i_p;k_p;\ldots;k_{d-1}} v_{i_q;k_q;\ldots;k_{d-1}})^2 + \theta \sum_{k_p;\ldots;k_q;\ldots k_{d-1}} \varsigma^2_{k_p;\ldots;k_q;\ldots;k_{d-1}} \quad (3.5)$$

that means an evaluation of the arguments ς of minimal energy, $1 >> \theta \geq 0$ – a parameter of the regularization.

Two following examples are used to show the properties of solutions of problem (3.5) in the case $d = 3$. The tensor is presentedin the form of 3-SVD (2.4):

$$x_{i_0;i_1;i_2} = \sum_{k_2} s_{k_2} v_{i_2;k_2} \sum_{k_1} u_{i_0;k_0;k_2} s_{k_1;k_2} v_{i_1;k_1;k_2} \quad (3.6)$$

Example 1. The projection

$$x_{i_1;i_2} = \sum_{k_1;k_2} \varsigma_{k_1;k_2} v_{i_1;k_1;k_2} v_{i_2;k_2} \quad (3.7)$$

of tensor' elements $x_{i_0;i_1;i_3}$ in dimensions 1×2 is defining by optimization of the functional (3.5) with using *3*-SVD (2.4). The condition of optimality (3.5) looks as

$$\frac{\partial P_\varsigma(i_1,i_2)}{\partial \varsigma_{q_1;q_2}} = \sum_{i_0;k_2;k_1} u_{i_0;k_1;k_2} s_{k_1;k_2} s_{k_2} \sum_{i_1} v_{i_1;k_1;k_2} v_{i_1;q_1;q_2} \cdot \cdot \sum_{i_2} v_{i_2;k_2} v_{i_2;q_2} - N_0 \sum_{k_2;k_1} \varsigma_{k_1;k_2} \sum_{i_1} v_{i_1;k_1;k_2} v_{i_1;q_1;q_2} \sum_{i_2} v_{i_2;k_2} v_{i_2;q_2} + \theta \cdot \varsigma_{q_1;q_2} = 0$$

The sum by i_2 is equal to $\delta_{k_2;q_2}$, so the sum by i_1 $\sum_{i_1} v_{i_1;k_1;k_2} v_{i_1;q_1;k_2} = \delta_{k_1;q_1}$. Therefore

$$\varsigma_{q_1;q_2} = \left(N_0 - \theta\right)^{-1} s_{q_1;q_2} s_{q_2} \sum_{i_0} u_{i_0;q_1;q_2} \quad (3.8)$$

The substituting the coefficients into (3.7) yields the projection

$$x_{i_1,i_2} = N_0^{-1} \sum_{k_1;k_2} s_{k_1;k_2} s_{k_2} v_{i_1;k_1;k_2} v_{i_2;k_2} \sum_{i_0} u_{i_0;k_1;k_2} \quad (3.9)$$

As it follows from (3.8), the minimal energy will be at $\theta = 0$. Projection (3.9) is equal to that which can be obtained by averaging the tensor elements (3.6) by index i_0 when are using all SVs.

Example 2. The optimization of functional (3.5) for the projection in dimensions 0×1

$$x_{i_0;i_1} = \sum_{k_1;k_2} \varsigma_{k_1;k_2} u_{i_0;k_1;k_2} v_{i_1;k_1;k_2} \quad (3.10)$$

gives the equations system

$$\sum_{k_1;k_2} \varsigma_{k_1;k_2} (N_2 A_{k_1;k_2;q_1;q_2} - \theta \cdot \delta_{k_1;q_1} \delta_{k_2;q_2}) = \sum_{k_1;k_2} s_{k_1;k_2} \sum_{i_2} s_{k_2} v_{i_2;k_2} A_{k_1;k_2;q_1;q_2} \quad (3.11)$$

where the tensor

$$A_{k_1;k_2;q_1;q_2} = \sum_{i_0} u_{i_0;k_1;k_2} u_{i_0;q_1;q_2} \sum_{i_1} v_{i_1;k_1;k_2} v_{i_1;q_1;q_2} \quad (3.12)$$

The vectors in (3.12) are mutually related because they are vectors of the SVDs of different vectors by indices k_2 which are orthogonal, so $\sum_{k_1;q_1} s_{k_1;k_2} A_{k_1;k_2;q_1;q_2} s_{q_1;q_2} = \delta_{k_2;q_2}$. Therefore system (3.11) is not of full rank. One of its solutions is the following.

$$\varsigma_{q_1;q_2} = N_2^{-1} s_{q_1;q_2} s_{q_2} \sum_{i_2} v_{i_2;q_2} \quad (3.13)$$

and the projection

$$x_{i_0;i_1} = N_2^{-1} \sum_{k_2;k_1} s_{k_1;k_2} s_{k_2} u_{i_0;k_1;k_2} v_{i_1;k_1;k_2} \sum_{i_2} v_{i_2;k_2} \quad (3.14)$$

is equal to projection which can be obtained by averaging by index i_2 in (3.6). The minimization of projection (3.10) energy can be made by iterative way with variation of θ in (3.11).

3.3. Singular projection

The dimension reducing and general projection are reduced to weighted and direct summations over eliminating dimensions. Unlike to conventional sum, projections (3.9) and (3.14) may be optimized in respect to energy by SP in a subspace of most significant vectors of the SVDs that correspond to the largest SVs in (3.8) and (3.13). Also, the projections allow to enforce main features of tensor' data. However, in this case, the SVs of those dimensions may be truncated which will not significantly impact on accuracy of the tensor representation. This is the distinct between using *d*-SVD and conventional averaging.

4. Optimized inverse features filters

A series of N_0 images feature fragments of size $N_1 \times N_2$ can be presented by a tensor of elements in *3*-SVD format

$$\varphi_{i_1;i_2;i_0} = \sum_{k_0} s_{k_0} v_{i_0;k_0} \sum_{k_1} u_{i_2;k_1;k_0} s_{k_1;k_0} v_{i_1;k_1;k_0} \quad (4.1)$$

as a result of projection operation (3.2), (3.4) or (3.9), (3.14). It can be reshaped as N_0 vectors of the size $N = N_1 N_2$: $\varphi_{i_1 i_2;i_0} = \varphi_{m;i}$. The problem of filters characteristics $f_{i;m}$ finding for images recognizing by their features can be formulated as an optimization of the functional

$$I_f = \underset{f}{argmin} \sum_{i;k} \left\{ \left| \sum_{m=0}^{N-1} f_{i;m} \varphi_{m;k} - \delta_{i;k} \right|^2 + \lambda \sum_{m=0}^{N-1} f_{i;m} f_{k;m} \right\} \quad (4.2)$$

where λ is a regularization parameter, $i,k = 0,\ldots,N_0 - 1$. The expression of the condition of optimality (4.2)

$$\frac{\partial I_f}{\partial f_{i;n}} = \sum_{i,k} \left(\left(\sum_{m=0}^{N-1} f_{i;m} \cdot \varphi_{m;k} - \delta_{i;k} \right) \cdot \varphi_{n;i} + \lambda \cdot \sum_{m=0}^{N-1} \delta_{i;n} \cdot f_{k;m} \right) = 0$$

in terms of the SVD

$$\varphi_{m;i} = \sum_{k=0}^{N_0-1} u_{m;k} s_k v_{i;k}; \quad f_{i;m} = \sum_{k=0}^{N_0-1} v_{i;k} \sigma_k u_{m;k} \quad (4.3)$$

can be transformed into the next one.

$$\sum_i \sum_{k=0}^{N_0-1} (\sigma_k s_k^2 - s_k) u_{n;i} v_{i;k} + \lambda \sum_i \sum_{k=0}^{N_0-1} \sigma_k u_{n;k} v_{i;k} = 0$$

from which it follows the known [9] ratio of SVs in (4.3):

$$\sigma_k = \frac{s_k}{s_k^2 + \lambda} \tag{4.4}$$

The filters performance – resolution and insensitivity to image fragments variations, depends on the choice of λ [3]. The value of λ is found by optimization of the functional

$$I_\lambda = \underset{\lambda}{argmin}\left\{\sum_{i;k}\left|\sum_{m=0}^{N^2-1} f_{i;m}(\lambda)\varphi_{m;k} - \delta_{i;k}\right|^2 + \frac{\sum_{i;k}\left(\sum_{m=0}^{N^2-1} f_{i;m}(\lambda) f_{k;m}(\lambda)\right)^2}{\sum_{i;k}\left(\sum_{m=0}^{N^2-1} f_{i;m}(0) f_{k;m}(0)\right)^2}\right\} \tag{4.5}$$

where $f_{i;m}(\lambda)$ are the perturbated by regularization filters characteristics. The using (4.3), (4.4), the orthogonality of the SVD' (4.3) vectors in the condition of optimality $\partial I_\lambda / \partial\lambda = 0$ yields that

$$\lambda' = \frac{\sum_{n=0}^{N_0-1}\frac{s_n^4}{\left(s_n^2+\lambda\right)^5}}{\sum_{n=0}^{N_0-1}\frac{s_n^2}{\left(s_n^2+\lambda\right)^3}\sum_{n=0}^{N_0-1}\frac{1}{s_n^4}} \tag{4.6}$$

An assessment of λ can be found numerically by iterative way up to the condition $|\lambda' - \lambda| << \lambda$ will be met. It can be used the initial value $\lambda = 0.005\, s_0$. The meaning of functional (4.5) is to balance the energy of resolution blurring and the filter energy change caused by regularization. As it follows from (4.2) and (4.5), the matrix of filters elements is inverse in respect to matrix of features elements. Therefore, the filters are defined as Inverse Feature Filters.

5. Numerical investigation

Analysis of an implementation of the *d*-SVD to feature extraction with projection in vector subspace of chosen dimensions was made with using the test array of 2600 images of 35 license plates characters with the size from 13×13 to 41×41 pixels, which are cropped from car images and are mutually slightly different. The array contains up to 200 thousand mutually shifted fragments of the images of the size 13×13. These fragments were used to find feature fragments by their averaging for each character. The aim is to obtain the features that allow to recognize as wide a range of target objects as possible with a minimum number of errors.

The feature fragments are extracted during a training process by accumulation of those fragments that give true recognitions of the objects by the filters in (4.3) which were created at the previous stage of the training. The accumulated fragments are used to evaluate the filters for the next training stage. The accumulated fragments should be transformed in such a way as to highlight the main characteristic features of the objects and thereby ensure the selectivity and range of capture of the filters. The following conditions of SP in *3*-SVD (4.1) subspace of accumulated fragments were investigated in the process of training and objects recognizing:

1) $s_{k_0} s_{k_1;k_0} > 0.01 \cdot s_0 s_{0;k_0}$
2) $s_{k_1;k_0} > 0.01 \cdot s_{0;k_0}$
3) $s_{k_1;k_0} > 0.1 \cdot s_{0;k_0}$
4) the projection with optimized on energy coefficients given by equation (3.11).

The results of recognizing are presented in table 1. The methods of errors elimination considered in [3] were used, such as filtration of false fragments, spectrums combination and amplitude selection. The results have shown that even a weak truncation of the SVs by the first condition considering values of s_{k_0} leads to a loss of information about the features of objects and the training process reaches a certain final level of recognizing accuracy. The use of strong truncation of approximately half of SVs $s_{k_1;k_0}$ along one dimension according to the third condition made it possible to extract the features of objects in such a way that the range of objects capture was expanded by a quarter in comparison with weak truncation by the second condition without loss of relative recognition accuracy. The application of the fourth condition gives an acceptable result, but in this case, it is inferior to the previous two. In the considered case, tensor (3.12), reshaped as a matrix of size $N_0 N_1 \times N_0 N_2$, has rank $N_1 N_2$. The assessment of regularization parameter (4.6) has a fast convergence in several iterations, $\lambda \sim 0.01 \cdot s_0$.

Table 1. Dynamic of training process

Training step	Projection condition	Fragments			True rating
		Feature	True	False	
0	1	1699	1055	644	0.6209
	2	1103	889	214	0.8059
	3	1280	1037	243	0.8101
	4	1262	1032	230	0.8177
20	1	1245	925	320	0.7429
	2	1004	893	111	0.8894
	3	1349	1192	157	0.8836
	4	1121	1021	100	0.9108
40	1	1175	960	215	0.8170
	2	930	851	79	0.9150
	3	1287	1179	108	0.9160
	4	980	896	84	0.9142
60	1	1097	912	185	0.8313
	2	919	852	67	0.9271
	3	1136	1047	89	0.9216
	4	982	891	91	0.9073
80	1	1082	891	191	0.8234
	2	912	843	69	0.9243
	3	1150	1064	86	0.9252
	4	957	873	84	0.9122
100	1	1076	897	179	0.8336
	2	919	868	51	0.9445
	3	1141	1083	58	0.9491
	4	936	864	72	0.9230

6. Conclusion

It can be made the conclusion basing on the test experiment. The offered technique of HOSVD in the form of *d*-SVD (2.4) makes it possible to optimize the structure of multidimensional data by selective SP in the vector spaces of the *d*-SVD. It differs from the known ones (1.4) and (1.7) by a smaller amount of calculations and the separation of the set of SVs by dimensions. The selective SP by dimensions can gives positive results in processing of multidimensional signals and big data by extraction their features.

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