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OPTIMIZING PARAMETERS FOR 4D HYPERCHAOTIC SYSTEM USING WALRUS OPTIMIZER ALGORITHM

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Abstract*. The walrus optimization Algorithm's (WaOA) crucial significance in improving and creating a hyperchaotic system is the main topic of this study. We have enlarged the six-term, three-dimensional chaotic Liu system to a four-dimensional system with seven terms. In order to create hyperchaotic behaviour with great efficiency, the WaOA algorithm is utilized to optimize the system parameters in order to maximize the biggest Lyapunov exponent. The system's dynamic features, such as the study of equilibrium points, the Jacobian matrix, Lyapunov exponent, coexistence, and the Lyapunov dimension (Kaplan-Yorke), have all been fully examined. Simulations of the suggested system utilizing NI Multisim (version 14.2) for electrical circuit simulation have shown the algorithm's efficacy. The work demonstrates the WaOA algorithm's noteworthy contribution to enhancing hyperchaotic systems performance and broadening their useful applications in a variety of domains.*

Keywords: metaheuristic, walrus optimizer algorithm, hyperchaotic system, circuit implementation

OPTYMALIZACJA PARAMETRÓW DLA SYSTEMU HIPERCHAOTYCZNEGO 4D Z WYKORZYSTANIEM ALGORYTMU OPTYMALIZACYJNEGO WALRUS

Streszczenie*. Głównym tematem niniejszego badania jest kluczowe znaczenie algorytmu optymalizacji walrus (WaOA) w ulepszaniu i tworzeniu systemów hiperchaotycznych. Rozszerzyliśmy sześcioelementowy, trójwymiarowy chaotyczny układ Liu do układu czterowymiarowego zawierającego siedem elementów. Aby z dużą skutecznością wywołać zachowanie hiperchaotyczne, zastosowano algorytm WaOA do optymalizacji parametrów układu w celu maksymalizacji największego wykładnika Lapunowa. W pełni zbadano cechy dynamiczne systemu, takie jak punkty równowagi, macierz jakobianowa, wykładnik Lapunowa, współistnienie oraz wymiar Lapunowa (Kaplan-Yorke). Symulacje proponowanego systemu przy użyciu oprogramowania NI Multisim (wersja 14.2) do symulacji obwodów elektrycznych wykazały skuteczność algorytmu. Praca pokazuje znaczący wkład algorytmu WaOA w poprawę wydajności systemów hiperchaotycznych i poszerzenie ich użytecznych zastosowań w różnych dziedzinach.*

Słowa kluczowe: metaheurystyka, algorytm optymalizacyjny walrus, system hiperchaotyczny, implementacja obwodu

Introduction

The foundation of mathematical and physics study is dynamic systems, which provide deep insights into the intricate behaviours seen in computer science, engineering, and nature. His topic began with the study of stability analysis and differential equations, and it has since expanded to include systems with progressively higher levels of complexity and wide-ranging applications in many others fields. Advances in computing tools have allowed researchers to investigate higher-dimensional systems, extending the bounds of chaos and broadening its applicability, whereas early studies mostly concentrated on understanding and paths of simple systems. Notably a more thorough framework for comprehending chaotic behaviour has been made available by the shift from studying three-dimensional systems to four-dimensional ones. These developments have important real-world uses, such as encryption, signal processing and secure communications, and they are not only theoretical exercises. In order to improve dynamic features like hyperchaos which are essential for applications needing high sensitivity to beginning circumstance and unpredictability – for example, researchers have extended classical chaotic systems, such Lornez and Rossler systems, to larger dimensions [3, 6, 7, 17, 18, 21, 25, 28, 30, 31].

Recent research has highlighted the close connection between algorithm and dynamic systems chaotic [1, 20], demonstrating how optimization techniques may identify optimal parameter values that enhance chaotic behaviour. Algorithm such as particle swarm optimization, Genetic algorithms, and the more recent "walrus Optimization Algorithm" WaOA have shown encouraging potential in this area. Though repeated optimization of parameter values to enhance chaos, these methods generate systems with distinct dynamic properties that are appropriate for specific application needs. By examining the complex parameter spaces of higher-dimensional dynamic systems, the WaOA technique, for instance, has shown itself to be extremely effective at optimizing Lyapunov exponents (LE_s). It is therefore a useful instrument for increasing the required dynamic attributes [5, 9, 10, 11, 22, 23].

The objective this research is to bridge the gap between the practical implementation of optimization techniques and the theoretical study of dynamical systems. The goal of the work is to systematically expand a three-dimensional state and improve its properties using the WaOA algorithms in order to produce a highly chaotic state with superior (LE_S) value. These efforts are crucial for a boarder range of practical applications, such as secure data transfer, encryption, and other fields that rely on unpredictability, in addition to advancing our scientific understanding of chaotic systems [4, 14].

By holistically integrating historical perspectives, theoretical, advancements, and cutting-edge algorithmic techniques, this proposal aims to drastically transform the domains of dynamic systems chaos theory. Both more recent advancements in dynamical systems optimization techniques and classical works, such Lorenz's ground-breaking research on chaotic systems (1963) [13], served as its foundation. Study makes the, methodology creative protentional possible by providing the groundwork for the incorporation of algorithms like WaOA with extensions to dynamical systems [22, 24, 27].

A key step in a successful hyperchaotic process, parameter estimation has generated a lot of attention in the field of chaotic control research. In recent years, a novel approach to optimization algorithms known as the Walrus Optimizer has emerged. It was developed by Mir Jalili and his team and strikes a compromise between efficiency, flexibility, and exploration. It might be a powerful tool for solving difficult issues in engineering, machine learning, and other fields where optimization is essential. This is done to turn the dynamic system into a hyperchaotic state and obtain the most accurate and optimal results [29].

The following succinctly describes this work's essential contribution:

- There are two positive Lyapunov exponents in the suggested system, namely (n-2)+ve LE_s.
- Based on a single parameter, this system (10) falls into the category of self-excited or concealed attractors.
- The WaOA algorithm was applied to the proposed dynamic system to find the best parameters that made the system hyperchaotic.
- Two saddle foci and saddle unstable points are present in the suggested system; an electronic circuit is created, demonstrating the system's effectiveness.

The article is structured into eight segments. Section 2 introduces the proposed algorithm, followed by Section 3, which outlines the suggested method. Section 4 provides a detailed description of the constructed 4D hyperchaotic system, while Section 5 explains the dynamics of the system. Section 6 demonstrates the application of the proposed system through the design of an electronic circuit, implemented using Multisim 14.2. Section 7 presents the results, and finally, Section 8 concludes the paper with a summary.

1. Walrus optimizer algorithm

Metaheuristic algorithms are intelligent optimization methods that guide the search process by balancing exploitation and exploration. The growing complexity of real-world optimization problems has driven the development of more metaheuristic algorithms. Accordingly, used a novel swarm intelligence algorithm called the Walrus optimizer, inspired by walrus behaviours, including migration, breeding, roosting, feeding, gathering, and evading based on key signals like (danger signals and safety signals). Where results show that the proposed algorithm offers distinctive stability features and highly competitive performance in handling high-dimensional benchmarks and real-world problems. The proposed algorithms approach supports the ongoing advancement and expanded application of artificial intelligence, enhances optimization efficiency, and provides powerful tools for solving complex real-world challenges.

In [24], the Walrus Optimization Algorithm was taken in its original form. It consists of (3) separate stages, inspired by the behaviour of the walrus:

1. Leading individuals to feeding grounds.
2. Migration of walruses to rocky shores.
3. Fighting or fleeing predators.

Mathematically, the walruses represent the members of the research which constitutes the population of walrus optimizer algorithm, and specifically, each walrus is a potential solution to the desired optimization problem and is organized as a vector of (m) variables, where (M) denotes the dimensionality of the problem. Thus, a matrix (X) of dimension (N) times (M) represents a set of (N) walruses.

$$\boldsymbol{X} = \begin{bmatrix} \boldsymbol{X_1} \\ \vdots \\ \boldsymbol{X_i} \\ \vdots \\ \boldsymbol{X_N} \end{bmatrix}_{N*M} = \begin{bmatrix} \boldsymbol{X_{1,1}} & \cdots & \boldsymbol{X_{1,j}} & \cdots & \boldsymbol{X_{1,m}} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ \boldsymbol{X_{i,1}} & \cdots & \boldsymbol{X_{i,j}} & \cdots & \boldsymbol{X_{i,m}} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ \boldsymbol{X_{N,1}} & \cdots & \boldsymbol{X_{N,j}} & \cdots & \boldsymbol{X_{N,M}} \end{bmatrix}_{N*m} \tag{1}$$

In this context, $\boldsymbol{X_i}$ denotes the ($\boldsymbol{i^{th}}$) walrus in the population, while $\boldsymbol{X_{i,j}}$ denotes the value of the ($\boldsymbol{j^{th}}$) optimization variable proposed by the ($\boldsymbol{i^{th}}$) walrus. The effectiveness of each potential solution (any walrus in the population) is assessed based on the fitness function value, given by ($\boldsymbol{F_i = F(x_i)}$).

$$\boldsymbol{F} = \begin{bmatrix} \boldsymbol{F_1} \\ \vdots \\ \boldsymbol{F_i} \\ \vdots \\ \boldsymbol{F_N} \end{bmatrix} = \begin{bmatrix} \boldsymbol{F(X_1)} \\ \vdots \\ \boldsymbol{F(X_i)} \\ \vdots \\ \boldsymbol{F(X_N)} \end{bmatrix}_{N*1} \tag{2}$$

where the fitness function is denoted by $\boldsymbol{F}$.

The lower bound (*LB*) and upper bound (*UB*) of the optimization variable are randomly assigned to the initial population of walruses prior to the start of the iteration procedure.

$$\boldsymbol{X_i = LB + rand\,.\,(UB - LB)} \tag{3}$$

Here, where "rand" represents a vector of uniformly distributed random real numbers between 0 and 1.

The algorithm's first phase represents the global search of the entire search space, known as exploration. This phase is referred to as the feeding strategy. In line with walrus behaviour, the strongest walrus guides the other walruses in the group toward food sources. The strongest, or best walrus is identified as the one with the lowest objective function value. Mathematically, this phase is modelled by updating the position of each walrus $\boldsymbol{X_i^{P_i}}$ as follows:

$$\boldsymbol{X_{i,j}^{P_1} = x_{i,j} + rand_{i,j}\,.\,(SW_j - I_{i,j}\,.\,x_{i,j})} \tag{4}$$

Here, $\boldsymbol{X_{i,j}^{P_i}}$ represent the (jth) dimension of the newly generated position of the (ith) walrus, SW denoted the best walrus in the population, and $\boldsymbol{I_{i,j}}$ are integer values of either (1) or (2). If an improved fitness function value is achieved, the old position is replaced by the newly generated one, as follows:

$$\boldsymbol{X_i} = \begin{cases} \boldsymbol{x_{i,j} + rand_{i,j}\,.\,(x_{i,k} - I_{i,j}\,.\,x_{i,j})}, & \boldsymbol{F_k < F_i} \\ \boldsymbol{x_{i,j}\,.\,(x_{i,j} - x_{k,j})}, & else \end{cases} \tag{5}$$

The walrus's movement to stony beaches or outcrops as a result of later summer air temperature increases is described in the second phase. Each walrus is said to go toward a randomly chosen walrus from the population during period. The following is the mathematical method used to update each walrus's location, $\boldsymbol{X_i^{P_2}}$.

$$\boldsymbol{X_i^{P_2}} = \begin{cases} \boldsymbol{X_i^{P_1}}, & \boldsymbol{F_i^{P_1} < F_i} \\ \boldsymbol{X_i}, & else \end{cases} \tag{6}$$

In the ($\boldsymbol{3^{rd}}$) phase, the escape and defensive actions of the against predators are represented. The strategy for evading and confronting predators involves the walruses shifting their position, but only within the vicinity of their previous location. The following formula may be used to get each walrus's newly produced location, $\boldsymbol{X_i^{P_3}}$:

$$\boldsymbol{X_{i,j}^{P_3} = x_{i,j} + \frac{LB_j}{t} + \frac{UB_j}{t} - rand\,.\,\frac{LB_j}{t}} \tag{7}$$

In this case, the current iteration number is denoted by (*t*). Additionally, it is updated as follows if the new position is superior than the old one:

$$\boldsymbol{X_i} = \begin{cases} \boldsymbol{X_i^{P_3}} & \boldsymbol{F_i^{P_3} < F_i} \\ \boldsymbol{X_i} & else \end{cases} \tag{8}$$

Upon completing the ($\boldsymbol{3^{rd}}$) phase, a single iteration of the WaOA is finalized. The iterative process continues until the predefined maximum number of iterations is reached. Ultimately, the optimal solution to the optimization problem is determined by the best walrus, meaning the one with the highest fitness function value. Figure 1 displays the WaOA implementation flowchart, and Algorithm 1 describes its pseudocode.

Algorithm 1. The pseudocode is given below to offer a summary of the steps of the (WaOA)

```
Input data related to the optimization problem
Specify the population size N and the maximum number
of iterations t_max.
Set the population's starting values at random
        for i = 1 to t_max
Depending on the value of an fitness function, update the strongest walrus
        for i =1 to N
Use the phase-one feeding approach
Compute the position X_i^P1
Update the location of each walrus
Apply phase 2 − Migration
Compute the position X_i^P2
Update the location of each walrus
Use step three, which involves avoiding and combating predators
Compute the position X_i^P3
Update the location of each walrus
end
Save the best walrus so far
end
The walrus with the highest fitness function value is the global optimum
solution
```

2. Suggested method

The study of chaotic systems has gained significant attention due to their unpredictable but deterministic nature. Traditional approaches to generating hyperchaotic systems typically involve increasing the complexity of the system, such as adding additional variables, nonlinear terms, or coupling mechanisms. However, achieving Hyperchaos through these traditional approaches can be computationally intensive and less than optimal. To solve this challenge, biologically inspires algorithms are crucial for accurately and effectively transforming chaotic systems into hyperchaotic ones. These algorithms focus on adjusting the parameters of chaotic systems to get the optimal values that integrate hyperchaotic behaviour. By analysing the systems dynamics and continuously adjusting the parameter, they aim to improve specific features, such as the Lyapunov exponent. This gauges the level of chaos. In this study, the Walrus Optimizer (WaOA) algorithms will be employed. This method employs a feed-forward techniques that consists of three stages: exploration, migration, and exploitation. Its primary objective is to navigate the large parameter space and identify values that enhance hyper-chaotic properties, such as raising the complexity of phase dynamics and chaos dimensionality.

These algorithm's performance will also be evaluated on a range of chaotic systems to ascertain how different factors affect the algorithm's dynamic behaviour. The ultimate objective is to determine the optimal parameter combinations that effectively transform conventional chaotic systems into hyperchaotic systems, increasing their complexity and boarding their applications in domains such as cryptography, secure communications, and nonlinear dynamics.

3. A new 4D hyperchaotic system

In 2013, Liu et al. [2, 12] presented a unique 3D system consisting of six terms, as shown below:

$$\begin{cases} \dot{x}_1 = a(x_2 - x_1) \\ \dot{x}_2 = -c + x_1 x_3 \\ \dot{x}_3 = b - x_2^2 \end{cases} \quad (9)$$

where x_1, x_2, x_3 represent the state variables, while a, b, c are the system parameters. This system exhibits chaotic attractor when $(a, b, c) = (1.5, 1.7, 0.05)$, whereas the corresponding LE_s as $(0.1928, 0.0002, -0.6443)$ and K-Y- dim $(D_L = 2.2992)$ [2]. By applying a coupling control strategy, a new 4D hyperchaotic system is constructed by combining the original system (9) with eq. ($\dot{x}_4 = -dx_1x_4$) as shown below:

$$\begin{cases} \dot{x}_1 = a(x_2 - x_1) \\ \dot{x}_2 = -c + x_1 x_3 \\ \dot{x}_3 = b - x_2^2 \\ \dot{x}_4 = -d x_1 x_4 \end{cases} \quad (10)$$

The variables x_1, x_2, x_3 and x_4 represent the state variables, with (d) being the parameter for coupling $(d \neq 0)$. Under the best parameters and (IC) given in (11), (12), this system produces hyperchaotic attractors, the related (LE_s) and Lyapunov dimensions are described in (13), (14), respectively. Fig. 1 shows the phase portraits of the suggest system.

$$(a, b, c, d) = \underbrace{(3.6702,\ 3.8970,\ 0.6249,\ 0.6114)}_{\text{Typical parameters}} \quad (11)$$

$$(x_o, y_o, z_o, w_o) = \underbrace{(0.1, 0.2, -0.25, 0.1)}_{IC} \quad (12)$$

$$\begin{cases} LE_1 = 0.8674 \\ LE_2 = 0.0027 \\ LE_3 = -\mathbf{0.0000} \\ LE_4 = -3.8378 \end{cases} \quad \sum_{i=1}^{4} LE_i = -2.9677 \quad (13)$$

$$D_{KY} = J + \frac{1}{LE_{i+1}} \sum_{i=1}^{J} LE_i \Rightarrow D_{KY} = 3 + \frac{0.8674}{|-3.8378|} = 3.2267 \quad (14)$$

Comparing the suggested system to other existing systems, Table 1 shows that it contains the fewest terms [8, 15, 16, 19, 26].

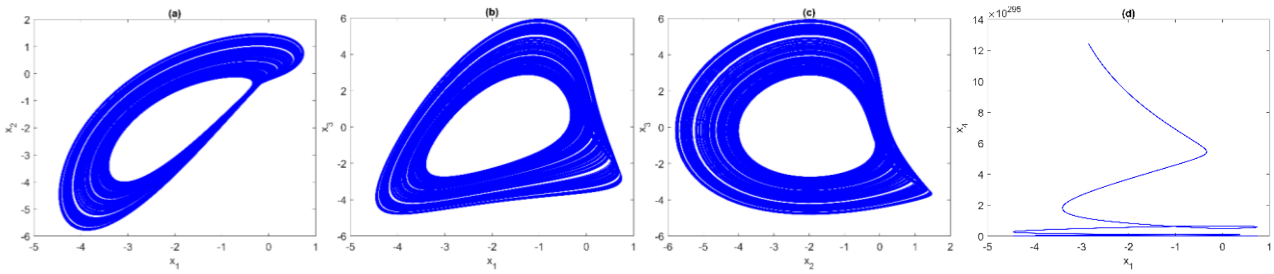

Fig. 1. Phase portraits of the system (10) in planes, (a) $x_1 - x_2$, (b) $x_1 - x_3$, (c) $x_2 - x_3$, (d) $x_1 - x_4$

Table 1. Comparison of some 4D mathematical models with the proposed model by algorithmic method

	Ref.	Equations	Method	Parameters	Total of terms	LE_S
i.	2022, [15]	$\begin{cases} \dot{x}_1 = a(x_2 + 0.2(x_1 - \varepsilon \sinh(x_1))) \\ \dot{x}_2 = bx_1 - x_2 + x_3 + x_4 \\ \dot{x}_3 = -c\,x_2 + x_4 \\ \dot{x}_4 = -dx_1 \end{cases}$	Bifurcation diagram	$\begin{cases} a = 8.15 \\ b = 0.8 \\ c = 12.5 \\ d = 0.5 \\ \varepsilon = 0.5 \end{cases}$	10-term	$(+,+,0,-)$
ii.	2020, [19]	$\begin{cases} \dot{x}_1 = a(x_2 - x_1) \\ \dot{x}_2 = -x_1x_3 + bx_2 - 5x_4 + 1 \\ \dot{x}_3 = x_1x_2 - cx_3 \\ \dot{x}_4 = dx_2 \end{cases}$	Bifurcation diagram	$\begin{cases} a = 30 \\ b = 20 \\ c = 3 \\ d = 0.1 \end{cases}$	9-term	$(+,+,0,-)$
iii.	2017, [26]	$\begin{cases} \dot{x}_1 = a(x_2 - x_1) \\ \dot{x}_2 = -bx_2 + nx_1x_3 + cx_4 \\ \dot{x}_3 = d - e^{x_1x_2} \\ \dot{x}_4 = -mx_2 \end{cases}$	Bifurcation diagram	$\begin{cases} a = 1 \\ b = 0.5 \\ c = 0.2 \\ d = 2.5 \\ n = 1 \\ m = 0.5 \end{cases}$	8-term	$(+,+,0,-)$
iv.	2022, [8]	$\begin{cases} \dot{x}_1 = a(x_2 - x_1) + kx_1x_3 + x_4 \\ \dot{x}_2 = -cx_2 - x_1x_3 \\ \dot{x}_3 = -b + x_1x_2 \\ \dot{x}_4 = -m\,x_2 \end{cases}$	Bifurcation diagram	$\begin{cases} a = 10 \\ b = 100 \\ c = 2.7 \\ k = -0.2 \\ m = 1 \end{cases}$	9-term	$(+,+,0,-)$
v.	2024, [16]	$\begin{cases} \dot{x}_1 = -24\,x_1 + 8x_2 \\ \dot{x}_2 = ax_1 + x_2 - 2x_1x_3 \\ \dot{x}_3 = bx_1x_2 - 4x_3 + x_4 \\ \dot{x}_4 = -x_1x_2 - 2x_3 - x_4 \end{cases}$	Bifurcation diagram	$\begin{cases} a = 20 \\ b = 1.1 \end{cases}$	11-term	$(+,+,0,-)$
vi.	This work	$\begin{cases} \dot{x}_1 = a(x_2 - x_1) \\ \dot{x}_2 = -c + x_1x_3 \\ \dot{x}_3 = b - x_2^2 \\ \dot{x}_4 = -dx_1x_4 \end{cases}$	By The Walrus Optimization Algorithm	$\begin{cases} a = 3.6702 \\ b = 3.8970 \\ c = 0.6249 \\ d = 0.6114 \end{cases}$	7-term	$(+,+,0,-)$

Comparing this work with several previous studies that relied on finding parameters randomly in order to achieve the principles of extreme chaos in a 4D system consisting of 7-terms. In contrast, this research proposed the (WaOA) algorithm, which was able to accurately find optimal parameters, where the proposed system, which presents a significant challenge in dynamic systems. This algorithm demonstrated great efficiency in achieving extreme chaos in this system, reflecting the progress made in parameter determination compared to previous works.

4. The dynamic properties

The Lyapunov exponent, equilibrium, dissipation, and bifurcation diagram are among the common dynamical phenomena of this system that are examined.

Dissipative

The divergence (trace of a Jacobian matrix) of the system (10) is computed as:

$$Tr\,(J) = \sum_{i=1}^{4} \frac{\partial \dot{x}_i}{\partial x_i} = -a - dx_1 \tag{15}$$

Equilibrium

Setting $\forall\, \dot{x}_i = 0$, yields two points that we may use to determine the equilibrium points for the suggest system:

$$E_{1,2} = \left(\pm\sqrt{b}, \pm\sqrt{b}, \pm\frac{c}{\sqrt{b}}, 0\right) \tag{16}$$

The Jacobian matrix of the new system at the points $E_{1,2}$ can be expressed as follows:

$$J = \begin{bmatrix} -a & a & 0 & 0 \\ x_3 & 0 & x_1 & 0 \\ 0 & -2x_2 & 0 & 0 \\ -dx_4 & 0 & 0 & -dx_1 \end{bmatrix} \Rightarrow$$

$$J(E_{1,2}) = \begin{bmatrix} -a & a & 0 & 0 \\ \pm\frac{c}{\sqrt{b}} & 0 & \pm\sqrt{b} & 0 \\ 0 & \pm 2\sqrt{b} & 0 & 0 \\ 0 & 0 & 0 & \mp d\sqrt{b} \end{bmatrix} \tag{17}$$

Eqs. (18) and (19), provide the characteristic equation and associated eigenvalues, respectively.

$$\lambda^4 + \underbrace{(a \pm d\sqrt{b})\lambda}_{p_1}{}^3 + \underbrace{\left(2b \pm ad\sqrt{b} \mp \frac{ac}{\sqrt{b}}\right)\lambda^2}_{p_2} + \underbrace{(2ab \pm 2bd\sqrt{b} - acd)}_{P_3}\lambda \pm \underbrace{2abd\sqrt{b}}_{P_4} = 0 \tag{18}$$

Eq. (18) produce the eigenvalues for both sites for the best parameters (11).

$$\underbrace{\begin{cases} \lambda_1 = 0.0987 + 2.7178i \\ \lambda_2 = 0.0987 - 2.7178i \\ \lambda_3 = -1.2070 + 0.0000i \\ \lambda_4 = -3.8677 + 0.0000i \end{cases}}_{E_1} \quad \underbrace{\begin{cases} \lambda_1 = 1.2070 + 0.0000i \\ \lambda_2 = -3.4669 + 0.0000i \\ \lambda_3 = -0.1016 + 2.8707i \\ \lambda_4 = -0.1016 - 2.8707i \end{cases}}_{E_2} \tag{19}$$

Therefore, both equilibrium $E_{1,2} = \left(\pm\sqrt{b}, \pm\sqrt{b}, \pm\frac{c}{\sqrt{b}}, 0\right)$ are unstable saddle foci and saddle points, respectively.

Lyapunov exponent

In attractors, to distinguish between chaotic and hyperchaotic systems, the Lyapunov exponent is the effective tool. For determining the LE_s, a variety of algorithms are available. The Wolf Algorithm is one of the most often used algorithms for calculating LE_s. With the parameters $(a,b,c,d) = (3.6702, 3.8970, 0.6249, 0.6114)$, step (sample time) = 0.25, observation period= 2500, and an IC (12), the LE_s of the new system is found numerically based on the trace of the Jacobian matrix (15). As illustrated in Fig. 2, this system has one nature: dissipative.

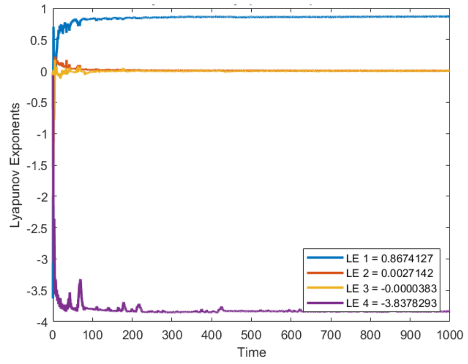

Fig. 2. Lyapunov exponents of the system (2) for (a,b,c) = (3.6702, 3.8970, 0.6249) *and d* = 0.6114

5. Circuit implementation

Hyperchaotic systems can be used to create many possible applications. As explained in this article, a readily modifiable analogue op-amp circuit is used to implement the system's state variable. The circuit equations for integral operation, inverse operation, and nonlinear product are carried out by electronic parts such as multipliers, capacitor, resistors, and operational amplifiers. A compliant analogue multiplier with an amplification factor of: V is the operational amplifier supply, and the parameters are set at $a = 3.6702$, $b = 3.8970$, $c = 0.6249$, $d = 0.6114$.

$$\begin{cases} \dot{x}_1 = -3.6702(x_2) - 3.6702(-x_1) \\ \dot{x}_2 = -0.6249 - (-x_1)(x_3) \\ \dot{x}_3 = -3.897(-v_0) - (-x_2)(x_2) \\ \dot{x}_4 = -0.6114(x_1)(x_4) \end{cases} \tag{20}$$

By utilizing Kirchhoff's law on the aforementioned system, we obtain

$$\begin{cases} \dot{x}_1 = -\frac{1}{R_1C_1}(x_1) - \frac{1}{R_2C_1}(x_2) \\ \dot{x}_2 = -\frac{1}{R_3C_2}(v_0) - \frac{1}{10R_4C_2}(-x_1)(x_3) \\ \dot{x}_3 = -\frac{1}{R_5C_3}(-v_0) - \frac{1}{10R_6C_3}(-x_2)(x_2) \\ \dot{x}_4 = -\frac{1}{10R_7C_4}(x_1)(x_4) \end{cases} \tag{21}$$

Select all the capacitors $(\forall C_i = 10\,nF,\ i = 1,2,3,4)$ and V_0 is 1 V. The related circuit parameters may be found in (21), Fig. 3, 4, which show a screenshot from Multisim 14.2. by comparing them to (19), (20).

$$\begin{cases} \frac{1}{R_3C_2} = 0.000006249 \Rightarrow R_3 = 160.0256\ k\Omega \\ \frac{1}{10R_4C_2} = 0.0001 \Rightarrow R_4 = 10\ k\Omega \\ \frac{1}{R_5C_3} = 0.00003897 \Rightarrow R_5 = 25.6607\ k\Omega \\ \frac{1}{10R_6C_3} = 0.0001 \Rightarrow R_6 = 10\ k\Omega \\ \frac{1}{10R_7C_4} = 0.00006114 \Rightarrow R_7 = 163.5590\ k\Omega \end{cases} \tag{22}$$

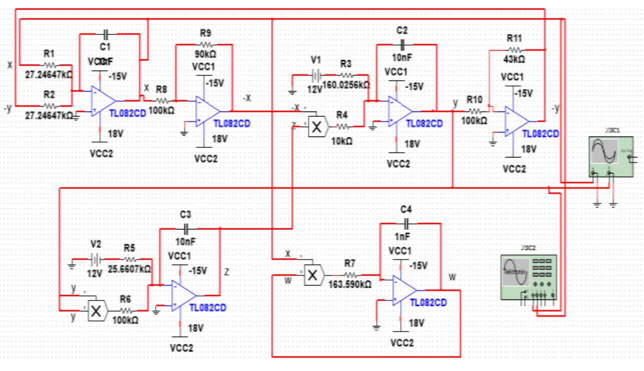

Fig. 3. Schematic of the system implementation circuit (10)

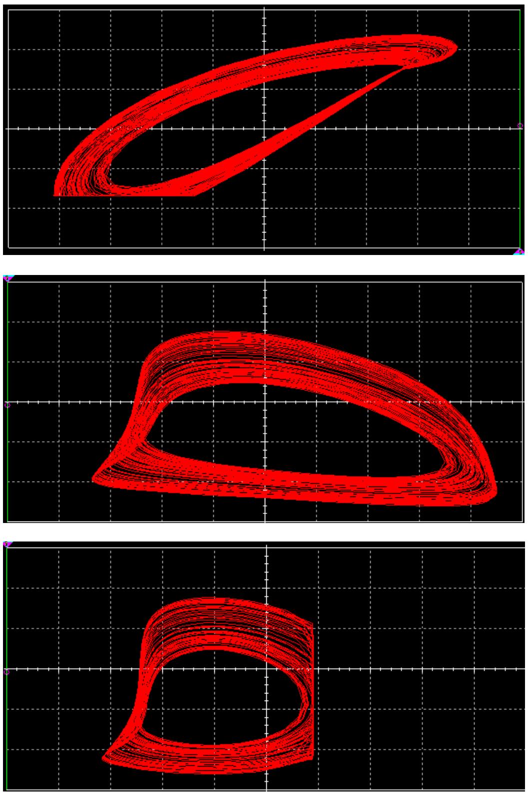

Fig. 4. Simulation results in Multisim $(a_1)\ x_1 - x_2$, $(a_2)\ x_1 - x_3$, $(a_3)\ x_2 - x_3$

5. The result

After the chaotic system parameters were estimated randomly to reach hyperchaotic system in this article, the Walrus Optimizer Algorithms (WaOA) was able to improve the largest Lyapunov exponent and increase its value from 0.45 to a wonderful and amazing value that reached (0.8674127). from estimating the hyper-chaotic system parameters scientifically, this exceptional improvement showed the accuracy and effectiveness of this algorithm used in determining the parameter spaces that work to increase chaos, as the results show the effectiveness and efficiency of the WaOA algorithm as a tool for examining and enhancing the complex dynamics of hyper-chaotic systems, which opens the way for more advanced applications and theoretical insights in the field of dynamic systems research.

6. Conclusions

A four-dimensional freestanding, hyperchaotic system modelled after the Liu system is presented in this work. It has two unstable foci saddle equilibrium points and the biggest Lyapunov exponent at their highest value. In specifically, the improved Walrus Optimizer (WaOA) is one of the sophisticated optimization methods that are included in this study. (WaOA) distinguishes by precisely estimating the system parameters to maximize the greatest Lyapunov exponent. The study emphasize how important optimization techniques are for investigation and improving the dynamic characteristics of complex systems. Coexistence phenomena, Lyapunov exponent, study equilibria and Lyapunov dimension (Kaplan-Yorke) are among the theoretical and numerical investigations that highlight the interaction between the algorithm and the hyperchaotic system. In addition. The results are validated by the electronic circuit's construction using NI Multisim, which closes the gap between theoretical modelling and practical implementations. These results highlight how optimization algorithm have the power to revolutionize our comprehension and application of dynamic systems.

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