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TOPOLOGICAL DERIVATIVE - THEORY AND APPLICATIONS

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Abstract. *The paper is devoted to present some mathematical aspects of the topological derivative and its applications in different fields of sciences such as shape optimization and inverse problems. First the definition of the topological derivative is given and the shape optimization problem is formulated. Next the form of the topological derivative is evaluated for a mixed boundary value problem defined in a geometrical domain. Finally, an example of an application of the topological derivative in the electric impedance tomography is presented.*

Keywords: Topological derivative, shape optimization, electrical impedance tomography

POCHODNA TOPOLOGICZNA – TEORIA I ZASTOSOWANIA

Streszczenie. *W pracy przedstawiono matematyczne aspekty dotyczące pochodnej topologicznej oraz jej zastosowań w różnych dziedzinach nauki, takich jak optymalizacja kształtu czy problemy odwrotne. W pierwszej części podano nieformalną definicję pochodnej topologicznej oraz sformułowano problem optymalizacji kształtu. Następnie wyprowadzono postać pochodnej topologicznej dla mieszanego problemu brzegowego. W ostatniej części przedstawiono przykład zastosowania pochodnej topologicznej dla problemu elektrycznej tomografii impedancyjnej.*

Słowa kluczowe: Pochodna topologiczna, optymalizacja kształtu, elektryczna tomografia impedancyjna.

Introduction

The Topological Derivative is defined as the first term of the asymptotic expansion of a given shape functional with respect to a small parameter that measures the size of singular domain perturbation [11, 12]. It represents the variation of the shape functional when the domain is perturbed by holes, inclusions, defects or cracks. The form of the Topological Derivative is obtained by the asymptotic analysis of a solution to elliptic boundary value problem in singularly perturbed domain combined with the asymptotic analysis of the shape functional all together with respect to the small parameter which measures the size of the perturbation. The definition of the Topological Derivative was introduced by Sokołowski and Zochowski in 1999 [13, 14]. Since then, the concept became extremely useful in the treatment of a wide range of problems [1, 3, 6]. Some tools of asymptotic analysis that allow to evaluate the form of the Topological Derivative was given in [8, 9].

Over the last decade, topological asymptotic analysis has become a broad, rich and fascinating research area from both theoretical and numerical standpoints. It has applications in many different fields [2, 5, 6], such as shape and topology optimization, inverse problems [3, 4], imaging processing [1], mechanical modeling including synthesis and/or optimal design of microstructures, fractures mechanics sensitivity analysis and damage evolution modeling.

1. Topological Derivative in Shape Optimization

The Topological Derivative evaluated for a given shape functional defined in a geometrical domain and dependent on a classical solution to elliptic boundary value problem is a principal tool in Shape Optimization. It represents the variation of the energy functional while the domain is singularly pertourbed by introducing a small hollow void.

Let $\Omega \subset \mathrm{R}^n$, $n = 2, 3$ an open set with $\Gamma = \partial\Omega$ local Lipschitz boundary of Ω, see Fig. 1 (left).

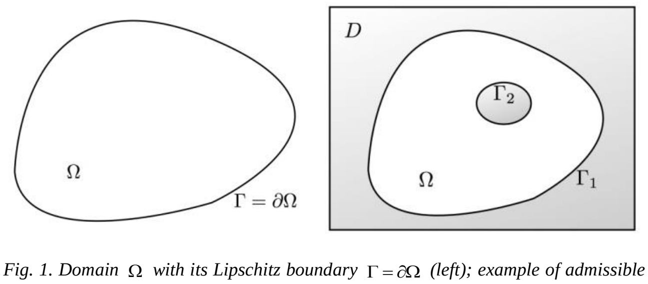

Fig. 1. Domain Ω with its Lipschitz boundary $\Gamma = \partial\Omega$ (left); example of admissible domain $\Omega \subset D$ (right)

Definition 1 *Let U_{ad} be a class of the admissible domains Ω in $\mathbf{R}^N$, $N = 2, 3$, with $\Omega \subset D$ where $D \subset \mathbf{R}^N$ is a hold-all domain.*

Shape optimization problem consist in finding a boundary Γ of the geometrical domain Ω which minimizes a given (shape) functional $\boldsymbol{J}(\Omega)$ (e.g. weigh of the structure), and subject to some supplementary conditions on volume, energy or displacement on the boundary. The conditions imposed on Ω can concern the following properties:

- volume: $\int_\Omega dx \le M, \quad M \in \mathbf{R}_+$,
- perimeter: $\int_{\partial\Omega} d\Gamma \le L, \quad L \in \mathbf{R}_+$
- regularity of the boundary - usually $\partial\Omega$ is locally Lipschitz.

1.1. Sobolev Spaces

Let us introduce some notations for the functional spaces, which are necessary for the analysis of shape optimization problems.

(i) We denote by $D(\Omega)$ a space of test functions in Ω. Thus, for an open set $\Omega \subset \mathrm{R}^n$, $n \ge 2$, $\varphi \in D(\Omega) \Rightarrow \varphi \in C_C^\infty(\Omega)$, is a smooth function with the compact support in Ω.

(ii) Denote by $D'(\Omega)$ space of distributions in Ω. Suppose that v is a funtion of $L^2(\Omega)$, it means that:

$$\int_\Omega |v(x)|^2 dx = \|v\|^2_{L^2(\Omega)} < \infty .$$

For $v \in D'(\Omega)$, we can define the operator $L_v(\varphi) = \int_\Omega v(x)\varphi(x), \quad \forall \varphi \in D(\Omega)$, the following definitione can be formulated:

Definition 2 *For the derivative in the sense of distributions* $\frac{\partial}{\partial x_i} v \in D'(\Omega)$ we have:

$$\left\langle \frac{\partial}{\partial x_i} v, \varphi \right\rangle_{D'(\Omega)\times D(\Omega)} = \int_\Omega v(x) \frac{\partial \varphi}{\partial x_i}(x) dx .$$

Remark. If $v \in C^1(\Omega)$ then $\frac{\partial}{\partial x_i} v = \frac{\partial v}{\partial x_i}$.

(iii) We recall the definition of the first order Sobolev Space. Denote by $H^1(\Omega) = \{v \in L^2(\Omega) : \frac{\partial}{\partial x_i} v \in L^2(\Omega), \quad i = 1, \dots, n\}$.

Scalar product in $H^1(\Omega)$ is defined as:

$$(u, v)_{H^1(\Omega)} = \int_\Omega \left[\frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_i} + uv \right] dx .$$

1.2. Problem formulation

The shape optimization problem is usually defined as a minimization of a given shape functional. The shape functional can be written as an integral over the domain Ω or on the boundary $\partial\Omega$ of a function which depends on a solution $u(\Omega;x)$ of some boundary value problem.

Suppose that the function $u(\Omega;x)\in H^1(\Omega),\ x\in\Omega$ is a solution to the following boundary value problem:

$$\begin{cases} -\Delta u(x) = f(x), & x\in\Omega \\ u(x) = g(x), & x\in\Gamma_D\subset\partial\Omega \\ \dfrac{\partial u}{\partial n}(x) = h(x), & x\in\Gamma_N\subset\partial\Omega \end{cases} \tag{1}$$

where $f\in L^2(\Omega)$, $g\in H^{1/2}(\Gamma_D)$, $h\in L^2(\Gamma_N)$ and $\partial\Omega=\Gamma_N\cup\Gamma_D$ boundary of the domain Ω. Consider the following shape functional:

$$\boldsymbol{J}(\Omega)=\int_\Omega J\left(x,u(\Omega;x)\right)dx\,, \tag{2}$$

where $J:\Omega\times\mathbb{R}\to\mathbb{R}$ is a function of class C^1 defined in $\Omega\times\mathbb{R}$ and depending on the solution $u(\Omega;x)$ to the Boundary Value Problem (1). The shape optimization problem can be written as:

$$\text{Find } \Omega^*\in U_{ad}\text{, such, that } \boldsymbol{J}(\Omega^*)=\inf_{\Omega\in U_{ad}}\boldsymbol{J}(\Omega)\,. \tag{3}$$

In order to decrease the values of the shape functional $\boldsymbol{J}(\Omega)$ we have two possibilities to change the shape of the domain Ω:

(a) deformation of the boundary of the domain produced by changes at the boundary governed by shape derivative (cf. Fig. 2 (left)).

(b) perforation of the domain by creating small holes inside Ω - topological changes governed by topological derivative (cf. Fig. 2 (right)).

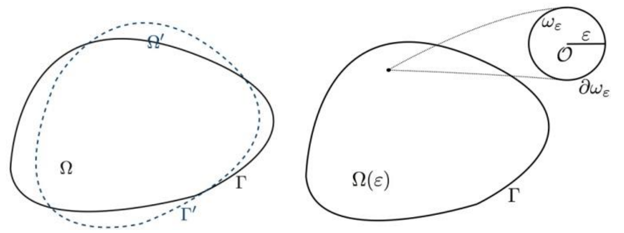

Fig. 2. Changes of the boundary (left) and changes of the topology (right) of the domain Ω

1.3. Topological derivative

Changes of the topology of the domain Ω are made by introducing a small hole inside Ω. Such new domain is called a singular geometrical perturbation of the domain Ω and is defined as $\Omega(\varepsilon)=\Omega\setminus\bar{\omega}_\varepsilon$, $\varepsilon>0$, where ω_ε is a (very small) subset of Ω created at point $\mathcal{O}$ (origin) and of size ε such that $\bar{\omega}_\varepsilon\subset\Omega$ [11, 13].

In the modified domain $\Omega(\varepsilon)$ (see Fig. 2 (right)) the boundary value problem (1) is redefined in the following way:

$$\begin{cases} -\Delta u_\varepsilon(x) = f(x), & x\in\Omega(\varepsilon), \\ u_\varepsilon(x) = g(x), & x\in\Gamma_D\subset\partial\Omega(\varepsilon) \\ \dfrac{\partial u_\varepsilon}{\partial n}(x) = h(x), & x\in\Gamma_N\subset\partial\Omega(\varepsilon) \\ \dfrac{\partial u_\varepsilon}{\partial n}(x) = 0, & x\in\partial\omega_\varepsilon \end{cases} \tag{4}$$

Note that in this problem, the boundary Γ is decomposed into two parts, one part is called Γ_D on which the Dirichlet boundary condition is imposed, and the other part Γ_N with the Neumann boundary condition.

Here, for $\varepsilon=0$ we have $\Omega(\varepsilon)=\Omega$ and we define a funtion of parameter $\varepsilon>0$:

$$j(\varepsilon)=\boldsymbol{J}(\Omega(\varepsilon))=\int_{\Omega(\varepsilon)}F(x,u_\varepsilon(\Omega(\varepsilon);x))dx \tag{5}$$

Analysis of the behavior of the function $j(\varepsilon)$ for $\varepsilon\to0^+$ allows us to establish the topological derivative $\Omega\ \mathcal{T}(\mathcal{O})$ of the functional $\boldsymbol{J}(\Omega)$. Using the asymptotic analysis [7, 8, 9] one can determine the form of the topological derivative $\mathcal{T}(\mathcal{O})$, $\mathcal{O}\in\Omega$ ($\mathcal{O}$ center of the hole ω_ε) of the solution $u_\varepsilon(\Omega;\chi)$ to the boundary value problem (4). If the value of the function $\mathcal{T}(\mathcal{O})$ is known $\mathcal{O}\in\Omega$, then we can expand the function j in the Taylor series and get the following equality:

$$j(\varepsilon)=j(0)+\varepsilon^2 j''(0^+)+O(\varepsilon^2) \tag{6}$$

and (6) can be rewritten in the following form:

$$\boldsymbol{J}(\Omega(\varepsilon))=\boldsymbol{J}(\Omega)+|\omega_\varepsilon|\mathcal{T}(\mathcal{O})+O(|\omega_\varepsilon|) \tag{7}$$

with $|\omega_\varepsilon|=\int_{\omega_\varepsilon}dx$ the volume of the hole ω_ε. Equality $\varepsilon^2 j''(0^+)=|\omega_\varepsilon|\mathcal{T}(\mathcal{O})$ takes place for the Neumann boundary conditions on $\partial\omega_\varepsilon$.

1.4. Shape derivative

Variations of the boundary $\Gamma=\partial\Omega$ are made using the so-called shape gradient. The shape gradient g_Γ is derived from shape derivative $dJ(\Omega;V)=\langle g_\Gamma,V\cdot n\rangle$ defined as follows [12]:

$$d\boldsymbol{J}(\Omega;V)=\lim_{t\to0}\frac{1}{t}\left(\boldsymbol{J}(\Omega_t)-\boldsymbol{J}(\Omega)\right). \tag{8}$$

The limit (8) is called derivative of the functional $\boldsymbol{J}(\Omega)$ in the direction of the vector field *V*. Domain Ω_t is a deformation of the domain Ω.

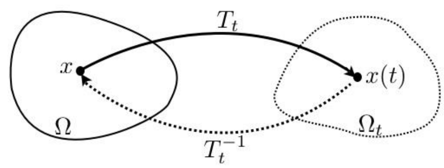

Fig. 3. Transformation of the domain Ω

This image Ω_t of Ω is obtained via the transformation $T_t:\mathbb{R}^2\to\mathbb{R}^2$ such that:

$$\Omega_t=T_t(\Omega),\Omega\in\mathbb{R}^N \tag{9}$$

Ω_t is a small perturbation of Ω and is defined as an image of Ω via the mapping T_t. The mapping is given by the flux of the vector field $V:\mathbb{R}\times\mathbb{R}^2\to\mathbb{R}^2$, which is defined as:

$$\begin{cases} \dfrac{dx}{dt}(t)=V(t,x(t)), \\ x(0)=X. \end{cases} \tag{10}$$

where $V(\cdot,\cdot)\in C^1\left([0,\delta);C^2(\mathbb{R}^N;\mathbb{R}^N)\right)$,

and $V(t,x(t))=\left(\dfrac{\partial T_t}{\partial t}\circ T_t^{-1}\right)(x)$.

Thus, in (10) we have the initial value $X\in\Omega\subset\mathbb{R}^2$ and a position of a particle $x(t)=T_t(X)\in\Omega_t$ such that we can write:

$$\Omega_t=\{x\in\mathbb{R}^2:\exists X\in\Omega\text{ such that }x=x(t,X)\} \tag{11}$$

In the expression $\langle g_\Gamma,V\cdot n\rangle$, the element g_Γ is a distribution with the support included in $\Gamma=\partial\Omega$ and $V\cdot n$ is a normal component of the vector field $V(0,\cdot)$ on the boundary Γ.

1.5. Algorithm for solving the standard shape optimization problem

The following algorithm can be applied in order to solve a traditional shape optimization problem. Here we present a scheme of the procedure that allows to find an optimal shape of a domain. Numerical method of finding a solution to a shape optimization problem which uses the topological derivative and the method of level set was presented in [2]. In [16] the numerical method of shape optimization for non-linear elliptic boundary value problem was described in details.

Step 1. Define a domain Ω for its shape optimization.

Step 2. Define a shape functional $J(\Omega;u)$ depending on the solution u to a boundary value problem given in the domain Ω.

Step 3. Solve the boundary value problem in the domain Ω.

Step 4. Create small holes in the domain Ω and/or deform the bounadry of the domain in order to minimize the shape function. In such modified domain solve the boundary value problem.

Step 5. Check the value of the shape function. If it is minimal, exit the process, if not return to Step 4.

2. Evaluation of the topological derivative for a mixed boundary value problem

The form of the topological derivative is obtained via asymptotic analysis of a boundary value problem and of an energy functional. The definition of the topological derivative was introduced in [13, 15]. Some notations of the asymptotic analysis that allow to evaluate the form of the topological derivative was given in [9, 10]. In this paper a mixed boundary value problem is considered, the domain decomposition is introduced in order to find the explicit form of the topological derivative.

2.1. Problem formulation

Let $\Omega\in\mathbb{R}^N$, N=2, 3 be a bounded domain with a smooth boundary $\Gamma=\partial\Omega$, see Fig. 4 (left).

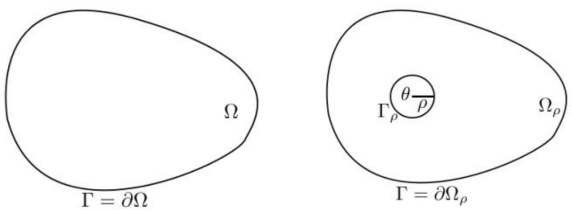

Fig. 4. Domain Ω with its smooth boundary $\Gamma=\partial\Omega$ (left) and its perturbation Ω_ρ (right)

In such domain we define the following elliptic boundary value problem:

$$\begin{cases} -\Delta u = f \quad \text{in } \Omega , \\ u=0 \quad \text{on } \Gamma=\partial\Omega . \end{cases} \tag{12}$$

with $f\in L^2(\Omega)$ where $L^2(\Omega)=\left\{ f:\int_\Omega |f|^2<\infty \right\}$. Next, we define the energy functional $\boldsymbol{J}:\Omega\longmapsto\mathbb{R}$ as:

$$\boldsymbol{J}(\Omega)=\frac{1}{2}\int_\Omega |\nabla u|^2-\int_\Omega fu \tag{13}$$

where u: $\Omega\longmapsto\mathbb{R}$ is a solution to the boundary value problem (12): Since $u\in H_0^1(\Omega)$ where $H_0^1(\Omega)=\left\{u\in H^1(\Omega):u=0 \quad \text{on} \quad \partial\Omega\right\}$ is a Hilbert space, then the elliptic equation can be written in the following variational formulation with a test function $\phi\in H_0^1(\Omega)$.

$$\int_\Omega \nabla u\nabla\phi=\int_\Omega f\phi, \quad \forall\phi\in H_0^1(\Omega) \tag{14}$$

Taking $\phi=u$ we get from (13):

$$\boldsymbol{J}(\Omega)=-\frac{1}{2}\int_\Omega |\nabla u|^2=-\frac{1}{2}\int_\Omega fu \tag{15}$$

Now, we modify the domain Ω by introducing a small hole $\omega_\rho=\left\{|x-\mathcal{O}|<\rho\right\}$ at point $\mathcal{O}$ and the radius ρ, see Fig. 4 (right). In such perturbed domain that we denote by $\Omega_\rho=\Omega\setminus\bar{\omega}_\rho$, the energy functional can be written in the form:

$$j(\rho)=\boldsymbol{J}(\Omega_\rho)=\frac{1}{2}\int_{\Omega_\rho}|\nabla u_\rho|^2-\int_{\Omega_\rho} fu_\rho, \quad \rho>0 \tag{16}$$

where u_ρ is a solution to a corresponding boundary value problem defined in the perturbed domain Ω_ρ. If $\rho=0$ then $\boldsymbol{J}(\Omega_\rho)\big|_{\rho=0}=\boldsymbol{J}(\Omega)$.

Depending on the type of boundary conditions imposed on the boundary Γ_ρ of a small inclusion ω_ρ, we can consider two cases of boundary value problems in perturbed domain Ω_ρ:

1. In the first case we suppose that the solution u_ρ is fixed on the boundary Γ_ρ of a small hole. Then, the corresponding Dirichlet boundary value problem defined in the domain Ω_ρ is the following:

$$\begin{cases} -\Delta u_\rho = f \quad \text{in } \Omega_\rho , \\ u_\rho=0 \quad \text{on } \Gamma\cup\Gamma_\rho . \end{cases} \tag{17}$$

In case of Dirichlet condition $u_\rho=0$ on the boundary Γ_ρ of a hole, for $\rho>0$ an asymptotic expansion of the energy functional $j(\rho)$ is the following:

$$j(\rho)=j(0)+\delta(\rho)\mathcal{T}_\rho(\Theta)+0(\delta(\rho)) \tag{18}$$

with

$$\delta(\rho)=\frac{1}{ln(\rho)} \; for \; \Omega\subset\mathbb{R}^2 \tag{19}$$

$$\delta(\rho)=\rho \; for \; \Omega\subset\mathbb{R}^3 \tag{20}$$

2. We can also suppose that the solution u_ρ is free on the boundary Γ_ρ of a small inclusion; thus the corresponding Neumann boundary value problem has the following form:

$$\begin{cases} -\Delta u_\rho = f \quad \text{in } \Omega_\rho , \\ u_\rho=0 \quad \text{on } \Gamma, \\ \dfrac{\partial u_\rho}{\partial n}=0 \quad \text{on } \Gamma_\rho. \end{cases} \tag{21}$$

For the Neumann condition $\partial_n u_\rho=0$ on the boundary Γ_ρ of a hole, we have:

$$j(\rho)=j(0)+\rho j'(0^+)+\frac{\delta(\rho)}{2}j''(0^+)+o(\rho^2) \tag{22}$$

with

$$\delta(\rho)=\rho^2 \; for \; \Omega\subset\mathbb{R}^2 \tag{23}$$

In the present paper we consider the second case, i.e. the mixed boundary value problem with Dirichlet boundary condition $u_\rho=0$ on the exterior boundary Γ and with Neumman boundary condition $\frac{\partial u_\rho}{\partial n}=0$ on the boundary Γ_ρ of a small hole.

2.2. Domain decomposition

According to [13], in order to get the form of the topological derivative we introduce the so-called domain decomposition of Ω_ρ by introducing a ring $C(R,\rho)=\{x\in\mathbb{R}^2:\rho<|x-\mathcal{O}|<R\}$, see Fig. 5, and we define the following sets Ω_R and Ω_ρ, cf. Fig. 6.

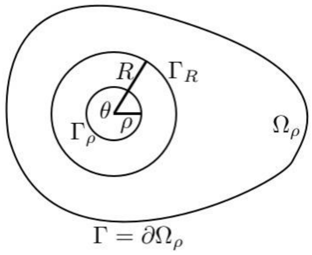

Fig. 5. Domain decomposition

$$\Omega_R = \Omega \setminus \overline{\{x \in \Omega : |x - \mathcal{O}| < R\}}, \qquad \Omega_\rho = \Omega \setminus \overline{\{x \in \Omega : |x - \mathcal{O}| < \rho\}}$$

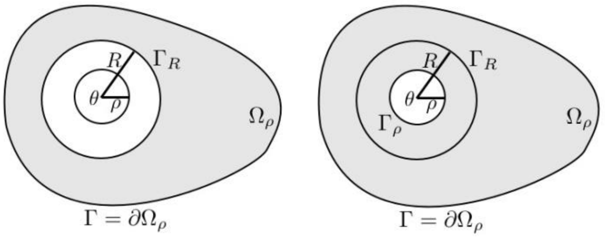

Fig. 6. Subdomain Ω_R (left) and Ω_ρ (right)

Note that $\Omega_\rho = \Omega_R \cup C(R,\rho)$.

Now we consider our problem in both subdomains Ω_R and $C(R,\rho)$. In Ω_R we have (cf. Fig. 7 (left)):

$$\begin{cases} -\Delta u_\rho = f\big|_{\Omega_R} & \text{in } \Omega_R, \\ u_R = 0 & \text{on } \Gamma, \\ \dfrac{\partial u_R}{\partial n} = A_\rho(u_R) & \text{on } \Gamma_R, \end{cases} \tag{25}$$

where $A_\rho : H^{(1/2)}(\Gamma_R) \ni v \mapsto \dfrac{\partial w_\rho}{\partial(-n)} \in H^{(-1/2)}(\Gamma_R)$ is a Steklov-Poincaré operator defining a flux on Γ_R.

In the ring $C(R,\rho)$ (cf. Fig. 7 (right)) we consider:

$$\begin{cases} -\Delta w_\rho = f\big|_{C(R,\rho)} & \text{in } C(R,\rho), \\ \dfrac{\partial w_\rho}{\partial n} = 0 & \text{on } \Gamma_\rho, \\ w_\rho = v & \text{on } \Gamma_R. \end{cases} \tag{26}$$

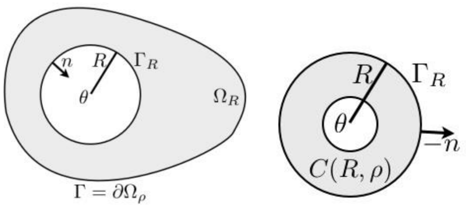

Fig. 7. Domain Ω_R (left) and the ring $C(R,\rho)$ (right)

Let us denote by $B_R(\mathcal{O}) = \{x \in \mathbb{R}^2 : |x - \mathcal{O}| \le R\}$ and suppose that $f\big|_{B_R(\Theta)} = 0$. Thus, in the ring $C(R,\rho)$ we consider the following boundary value problem:

$$\begin{cases} -\Delta w_\rho = 0 & \text{in } C(R,\rho), \\ \dfrac{\partial w_\rho}{\partial n} = 0 & \text{on } \Gamma_\rho, \\ w_\rho = v & \text{on } \Gamma_R. \end{cases} \tag{27}$$

By the Green formulae[1], the variational formulation of the problem is the following (taking w_ρ as a test function):

$$0 = -\int_{C(R,\rho)} \Delta w_\rho w_\rho = \int_{C(R,\rho)} \left|\nabla w_\rho\right|^2 - \left(\int_{\Gamma_R} \frac{\partial w_\rho}{\partial(-n)} w_\rho d\Gamma(x) + \int_{\Gamma_\rho} \frac{\partial w_\rho}{\partial n} w_\rho d\Gamma(x) \right) = (**) \tag{28}$$

We know that $\dfrac{\partial w_\rho}{\partial n} = 0$ on Γ_ρ and $w_\rho = v \in H^{1/2}(\Gamma_R)$ on Γ_R. Moreover, from the definition of the operator A_ρ we have $A_\rho = \dfrac{\partial w_\rho}{\partial(-n)}$, than:

$$(**) = \int_{C(R,\rho)} \left|\nabla w_\rho\right|^2 - \int_{\Gamma_R} v \cdot A_\rho(v) d\Gamma(x). \tag{29}$$

Thus:

$$\int_{C(R,\rho)} \left|\nabla w_\rho\right|^2 = \int_{\Gamma_R} v \cdot A_\rho(v) d\Gamma(x) = \left\langle A_\rho(v), v \right\rangle_{H^{(-1/2)}(\Gamma_R) \times H^{(1/2)}(\Gamma_R)}. \tag{30}$$

The term on the left hand side is the energy (if we know the asymptotic expansion of the energy functional with respect to ρ, then we know also the norm of A_ρ and the first term of the asymptotic expansion of the energy functional).

From (30) we have the following properties of the operator A_ρ:

- The operator A_ρ is positive:

$$\left\langle A_\rho(v), v \right\rangle = \int_{C(R,\rho)} \left|\nabla w_\rho\right|^2 > 0, \quad \forall v \ne 0 \tag{31}$$

- The operator A_ρ is linear and symmetric since:

$$\begin{aligned} \left\langle A_\rho(\varsigma), v \right\rangle_{H^{(-1/2)}(\Gamma_R) \times H^{(1/2)}(\Gamma_R)} &= \int_{C(R,\rho)} \nabla w_\rho(\varsigma) \nabla w_\rho(v) = \\ &= \int_{C(R,\rho)} \nabla w_\rho(v) \nabla w_\rho(\varsigma) = \left\langle v, A_\rho(\varsigma) \right\rangle \end{aligned} \tag{32}$$

Let us consider now the term $\int_{C(R,\rho)} \left|\nabla w_\rho\right|^2$. Let $v \in H^{(1/2)}(\Gamma_R)$ and take $x = r\cos\phi, \quad y = r\sin\phi$. Thus:

$$\Delta = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \phi^2} \tag{33}$$

and

$$\Gamma_R = \{(r,\phi) : r = R\}, \quad \Gamma_\rho = \{(r,\phi) : r = \rho\} \tag{34}$$

Assume that v has the following expansion of the Fourier series (it is the best approximation of the function v):

$$v(\phi) = \frac{1}{2}a_0 + \sum_{k=1}^{\infty} \left[a_k \sin(k\phi) + b_k \cos(k\phi) \right] \tag{35}$$

From the boundary condition on Γ_R in (27), we have that $v = w_\rho$, and since $w_\rho \in H^1(\Omega_R)$ $w_\rho\big|_{\Gamma_R} \in H^{(1/2)}(\Gamma_R)$ so $v \in H^{(1/2)}(\Gamma_R)$ and also $v \in H^{(1/2)}(\Omega_R)$. We can check easily that:

$$\|v\|^2_{H^1(\Gamma_R)} = \|v\|^2_{L^2(\Gamma_R)} + \|v'\|^2_{L^2(\Gamma_R)} \tag{36}$$

For $\|v\|^2_{L^2(\Gamma_R)}$ we have:

$$\begin{aligned} \|v\|^2_{L^2(\Gamma_R)} &= \int_{\Gamma_R} |v(\phi)|^2 d\phi = \frac{1}{2}a_0^2 \int_0^{2\pi} d\phi + \sum_{k=1}^{\infty} a_k^2 \int_0^{2\pi} \sin^2 k\phi + \sum_{k=1}^{\infty} b_k^2 \int_0^{2\pi} \cos^2 k\phi = \\ &= \pi a_0^2 + \pi \sum_{k=1}^{\infty} (a_k^2 + b_k^2) \le M \end{aligned} \tag{37}$$

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[1] Green formula: $0 = -\int_\Omega \Delta u v dx = \int_\Omega \nabla u \nabla v dx - \int_{\partial\Omega} v \frac{\partial u}{\partial n} d\Gamma(x) = \int_\Omega f v dx$, and from the assumption $f = 0$ in Ω.

For $\|v'\|^2_{L^2(\Gamma_R)}$, since $v'(\phi)=\sum_{k=1}^{\infty} k(a_k\cos k\phi - b_k\sin k\phi)$, we get:

$$\|v'(\phi)\|^2_{L^2(\Gamma_R)} = \int_{\Gamma_R} |v'(\phi)|^2 d\phi = \sum_{k=1}^{\infty} k^2 a_k^2 \int_0^{2\pi} \cos^2 k\phi + \sum_{k=1}^{\infty} k^2 b_k^2 \int_0^{2\pi} \sin^2 k\phi = \pi\sum_{k=1}^{\infty} k^2 (a_k^2+b_k^2) \le M \tag{38}$$

From the properties of Hilbert space $H^{(1/2)}(\Gamma_R)$ we have that:

$$\|v\|^2_{L^2(\Gamma_R)} = \sum_{k=1}^{\infty} \sqrt{1+k^2}\,(a_k^2+b_k^2) \tag{39}$$

and from (37) and (38) we get that $\|v\|^2_{L^2(\Gamma_R)} \le \boldsymbol{M}$

For $\rho=0$ we have:

$$C(R,0)=B(R)=\{x\in\mathbb{R}^2: |x-O|<R\}$$

Thus:

$$E(v)=\int_{B(R)} |\nabla w|^2 \tag{40}$$

with w the solution to the boundary value problem:

$$\begin{cases} -\Delta w = 0 & \text{in } B(R), \\ w = v & \text{on } \Gamma_R. \end{cases} \tag{41}$$

For $\rho>0$ then:

$$E_\rho(v)=\int_{C(R,\rho)} |\nabla w_\rho|^2 dx \tag{42}$$

with w_ρ a solution of the boundary value problem (27).

Let us now find the asymptotic expansion of the energy function (42) in order to determine the first term of the expansion. We suppose that the harmonic function w has the following Fourier series:

$$w=\frac{1}{2}a_0+\sum_{k=1}^{\infty}\left(\frac{r}{R}\right)^k (a_k\sin(k\phi)+b_k\cos(k\phi)) \tag{43}$$

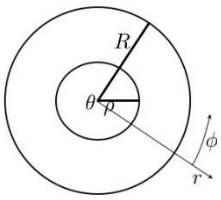

Fig. 8. Description of the ring in polar coordinates

For $r = R$ we have $w = v$ on the boundary Γ_R and our solution has the following form:

$$w_\rho=\frac{1}{2}c_{0,\rho}(r)+\sum_{k=1}^{\infty} c_{k,\rho}(r)(a_k\sin(k\phi)+b_k\cos(k\phi)) \tag{44}$$

where $c_{0,\rho}$ and $c_{k,\rho}$, $k\ge 0$ are the coefficients of the development. Since:

$$\Delta w_\rho=\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\phi^2}\right)w_\rho \tag{45}$$

then:

$$\Delta w_\rho=\frac{1}{2}c''_{0,\rho}(r)+\sum_{k=1}^{\infty}c''_{k,\rho}(r)(a_k\sin(k\phi)+b_k\cos(k\phi))+ \\ +\frac{1}{r}\frac{1}{2}c'_{0,\rho}(r)+\frac{1}{r}\sum_{k=1}^{\infty}c'_{k,\rho}(r)(a_k\sin(k\phi)+b_k\cos(k\phi))+ \\ -\frac{1}{r^2}\sum_{k=1}^{\infty}k^2 c_{k,\rho}(r)(a_k\sin(k\phi)+b_k\cos(k\phi))=0$$

and then:

$$\frac{1}{2}c''_{0,\rho}+\frac{1}{r}\frac{1}{2}c'_{0,\rho}(r)=0 \quad\Rightarrow\quad c_{0,\rho}(r)=\lambda\ln r+\mu$$

and for fixed k we have:

$$c''_{k,\rho}(r)+\frac{1}{r}c'_{k,\rho}(r)+\frac{1}{r^2}k^2 c'_{k,\rho}(r)=0 \quad\Rightarrow\quad c_{k,\rho}(r)=A_k r^k+B_k r^{-k},\quad k\ge 1$$

From the boundary conditons in (27) $w_\rho=0$ on Γ_ρ and $w_\rho=v$ on Γ_R we get:

- for $k = 0$

$$\lambda\ln R+\mu=a_0 \quad\text{on } \Gamma_R,\quad \lambda\ln\rho+\mu=a_0 \quad\text{on } \Gamma_\rho$$

Taking a sum of previous two equations we get:

$$\lambda\ln\frac{R}{\rho}=a_0 \quad\Rightarrow\quad \lambda=a_0\left(\ln\frac{R}{\rho}\right)^{-1} \quad\text{and}\quad \mu=-a_0\left(\ln\frac{R}{\rho}\right)^{-1}\ln\rho \tag{46}$$

and then:

$$c_{0,\rho}=a_0\left(\ln\frac{R}{\rho}\right)^{-1}\ln r-a_0\left(\ln\frac{R}{\rho}\right)^{-1}\ln\rho=\frac{a_0\ln\frac{r}{\rho}}{\ln\frac{R}{\rho}} \tag{47}$$

- for $k\ge 1$ we get (see (35)):

$$A_k R^k+B_k R^{-k}=1 \quad\text{on } \Gamma_R,\quad A_k R^k+B_k R^{-k}=0 \quad\text{on } \Gamma_\rho \tag{48}$$

and

$$A_k=\frac{R^k}{R^{2k}-\rho^{2k}} \qquad B_k=-A_k\rho^{2k} \tag{49}$$

so

$$c_{k,\rho}(r)=\left(\frac{r}{R}\right)^k-\frac{\rho^{2k}}{R^{2k}-\rho^{2k}}\left(\left(\frac{R}{r}\right)^k-\left(\frac{r}{R}\right)^k\right) \tag{50}$$

w_ρ can be written as $w_\rho=w+z_\rho$ where z_ρ is a perturbation of w with respect to ρ. Than:

$$z_\rho=\frac{1}{2}a_0\frac{\ln\frac{r}{R}}{\ln\frac{r}{\rho}}-\sum_{k=1}^{\infty}\frac{\rho^{2k}}{R^{2k}-\rho^{2k}}\left(\frac{R^k}{r^k}-\frac{r^k}{R^k}\right)(a_k\sin k\phi+b_k\cos k\phi) \tag{51}$$

for 0 < $0<\rho<R$. Returning to (42) we have:

$$E_\rho(v)=\int_{C(R,\rho)}|\nabla w_\rho|^2 dx=\int_{C(R,\rho)}|\nabla w+\nabla z_\rho|^2 dx \tag{52}$$

since $|\nabla w+\nabla z_\rho|^2=|\nabla w|^2+2\nabla w\nabla z_\rho+|\nabla z_\rho|^2$ then, using the formula $\nabla f=\begin{pmatrix} f|_r \\ \frac{1}{r}f|_\phi \end{pmatrix}$ and $|\nabla f|^2=f^2|_R+\frac{1}{r^2}f^2|_\phi$ we get:

$$E_\rho(v)=\int_{C(R,\rho)}|\nabla w_\rho|^2+2\int_{C(R,\rho)}\left(w|_r\, z_\rho|_r+\frac{1}{r^2}w|_\phi\, z_\rho|_\phi\right)+ \\ +\int_{C(R,\rho)}\left(z_\rho^2|_r+\frac{1}{r^2}z_\rho^2|_\phi\right) \tag{53}$$

Let us observe that the second integral in (53) disappear according to the ortogonality of polynomials and for the first integral we have

$$\int_{C(R,\rho)}|\nabla w_\rho|^2=\int_{B(R)}|\nabla w_\rho|^2-\int_{B(\rho)}|\nabla w_\rho|^2=E(v)-\int_{B(\rho)}|\nabla w_\rho|^2 \tag{54}$$

Let us consider the third integral, we get:

$$\int_{C(R,\rho)}\left(z_\rho^2|_r+\frac{1}{r^2}z_\rho^2|_\phi\right)=\int_0^{2\pi}\int_\rho^R\left(\frac{a_0}{2r\ln\frac{R}{\rho}}\right)^2 r\,dr\,d\theta+o(\rho)^2=\frac{a_0^2\pi}{2\ln\frac{R}{\rho}}+o(\rho)^2 \tag{55}$$

and since $\ln\frac{R}{\rho}=\ln R-\ln\rho$ then we get:

$$\int_{C(R,\rho)}\left(z_\rho^2|_r+\frac{1}{r^2}z_\rho^2|_\phi\right)=-\frac{a_0^2\pi}{2\ln\rho}+O\left(|\ln\rho|^{-2}\right).$$

Finally:

$$E_\rho(v) = E(v) - \frac{a_0^2\pi}{2\ln\rho} + O\left(\left|\ln\rho\right|^{-2}\right) \tag{56}$$

Thus we have the following theorem:

Theorem 1

$$E_\rho(v) = E(v) - \frac{a_0^2\pi}{2\ln\rho} + \Re(v) \tag{57}$$

the expression $\frac{a_0^2\pi}{2\ln\rho}$ *is the first term of the asymptotic expantion of the energy functional, and the remainder* $\Re(v)$ *is estimated by* $\left|\Re(v)\right| \le \frac{M}{\left|\ln\rho\right|^2}$ *and for* $\rho \to 0^+$ *we have* $\frac{1}{\left|\ln\rho\right|} \to 0$.

3. Applications

3.1. Example: Electrical Impedance Tomography

Electrical Impedance Tomography (EIT) is a non-destructive imaging technique which has various applications in medical imaging, geophysics and other fields.

Its purpose is to reconstruct the electric conductivity and permittivity of hidden objects inside a medium with the help of boundary field measurements. This part was prepared based on the papers of M. Hintermuller, A. Laurain and A. Novotny [3, 4].

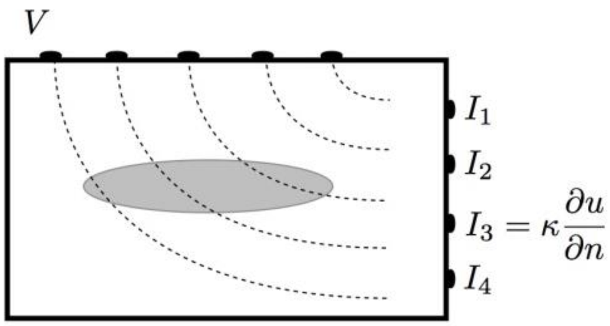

Fig. 9. Signal propagation in Electrical Impedance Tomography

Let us denote by Ω a bounded domain in $\mathbb{R}^N, N \ge 2$ being the background medium with Σ its smooth boundary where the currents are applied. Assume that Ω contains material with electrical conductivity $q(x) \ge q_0 > 0$. Then the electrical potential $u(x)$ satisfies:

$$-div\left(q\nabla u\right) = 0 \quad \text{in} \quad \Omega, \tag{58}$$

$$q\partial_n u = f \quad \text{on} \quad \Sigma, \tag{59}$$

where f is an applied current density on Σ satisfying the conservation of chargé $\int_\Sigma f(s)ds = 0$. The EIT problem consists in finding the electrical conductivity $q(x)$ inside Ω using a set of given values of applied current densities $f_i(x)$, i = 1, ..., M, on Σ and the corresponding electrical potentials $u_i(x)$ on Σ. Here we assume that the conductivity is piecewise constant and that it takes two distinct values, q_1 and q_2. Then, Ω can be split into two disjoint domains Ω_1 and Ω_2, with $\Omega = \Omega_1 \cup \Omega_2$ and conductivities q_1 and q_2, respectively, so that $\Sigma \cap \Gamma = \varnothing$ with $\Gamma = \partial\Omega_1$ (see Fig. 8). We then have $q = q_1\vec{l}_{\Omega_1} + q_2\vec{l}_{\Omega_2}$. Due to the particular form of q the regularization term becomes:

$$\int_\Omega \left|\nabla q\right| = \left|q_2 - q_1\right|\mathcal{P}(\Gamma), \tag{60}$$

where $\mathcal{P}(\Gamma)$ stands for the perimeter of Ω_1.

Therefore, the problem is reduced to solving the following problem which depends only on Ω_2 and the scalar values q_1, q_2:

$$\text{minimize} \;\; J(\Omega_2, q_1, q_2) = \sum_{i=1}^{M}\int_\Sigma (u_i - m_i)^2 + \beta\left|q_2 - q_1\right|\mathcal{P}(\Gamma), \tag{61}$$

where u_i is the solution of:

$$-div\left(q\nabla u_i\right) = 0 \quad \text{in} \quad \Omega, \tag{62}$$

$$q\partial_n u_i = f \quad \text{on} \quad \Sigma, \tag{63}$$

with f_i a known boundary current density for $i \in \{1, ..., M\}$.

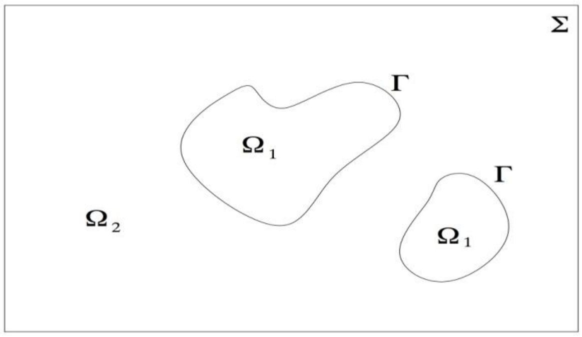

Fig. 10. Domain $\Omega = \Omega_1 \cup \Omega_2$

Further, m_i is the boundary measurement corresponding to f_i. In order to fulfill the compatibility conditions required for the Neumann boundary conditio (7), the measurements must satisfy:

$$\int_\Sigma m_i = 0, \quad i = 1, ..., M. \tag{64}$$

Since the solution of the Neumann problem (62)-(63) is not uniquely defined, we impose the condition:

$$\int_\Sigma u_i = 0, \quad i = 1, ..., M\;, \tag{65}$$

in order to obtain uniqueness. We also introduce the functional:

$$\boldsymbol{J}(\Omega_2) = \sum_{i=1}^{M}\int_\Sigma (u_i - m_i)^2 + \beta\left|q_2 - q_1\right|\mathcal{P}(\Gamma), \tag{66}$$

where q_1, q_2 are now assumed to be fixed. Referring back to (61) we clearly see that it contains Ω_2 as an unknown quantity. Hence, (61) represents a shape optimization problem.

3.2. Topological derivative

Now we assume that the domain Ω_1 is a small ball of radius ε and center $\hat{x} = \left(\hat{x}_1, \hat{x}_2\right) \in \Omega$, and we write Ω_1^ε instead of Ω_1. This allows to perform an asymptotic expansion of the shape functional $\boldsymbol{J}\left(\Omega_2^\varepsilon\right)$ with respect to ε. Here we use $\Omega_2^\varepsilon := \Omega \setminus \Omega_1^\varepsilon$. Eventually, this provides the topological derivative of $\boldsymbol{J}$. In what follows, we assume, for the sake of simplicity, that $\hat{x} = (0,0)$. We also introduce $\Gamma_\varepsilon := \partial\Omega_1^\varepsilon$. In this simplified framework, we are able to prove that the solution u_i can be written as $u_i = u_{1,i}^\varepsilon \vec{l}_{\Omega_1^\varepsilon} + u_{2,i}^\varepsilon \vec{l}_{\Omega_2^\varepsilon}$, with $\left(u_{1,i}^\varepsilon, u_{2,i}^\varepsilon\right)$ the solution of the coupled system:

$$-\Delta u_{2,i}^\varepsilon = 0 \;\; \text{in} \;\; \Omega_2^\varepsilon, \;\; q_2\partial_{n_2} u_{2,i}^\varepsilon = f_i \;\; \text{on} \;\; \Sigma \tag{67}$$

$$-\Delta u_{1,i}^\varepsilon = 0 \;\text{in}\; \Omega_1^\varepsilon, \; u_{1,i}^\varepsilon = u_{2,i}^\varepsilon \;\text{on}\; \Gamma_\varepsilon, \; q_2\partial_{n_1} u_{2,i}^\varepsilon = q_1\partial_{n_1} u_{1,i}^\varepsilon \;\text{on}\; \Gamma_\varepsilon \tag{68}$$

Here n_1 and n_2 stand for the outer unit normal vector to Ω_1^ε and Ω_2^ε, respectively. Thus, on Γ_ε, we have $n_1 = -n_2$. The normal derivatives with respect to n_1 and n_2 are $\partial_{n_1} = \nabla_x \cdot \mathrm{n}_1$ and $\partial_{n_2} = \nabla_x \cdot \mathrm{n}_2$. In what follows, for the sake of simplicity, we will drop the subscript i. This corresponds to only one measurement, i.e., $M = 1$. We point out that the case of several measurements is readily deduced from the case $M = 1$.

For the asymptotic expansion, we consider the following problems associated with (67)-(68):

$$-\Delta u_2^\varepsilon = 0 \text{ in } \Omega_2^\varepsilon,\; q_2\partial_{n_2} u_2^\varepsilon = f \text{ on } \Sigma,\; q_2\partial_{n_1} u_2^\varepsilon = q_1\partial_{n_1} u_1^\varepsilon \text{ on } \Gamma_\varepsilon, \quad (69)$$

$$-\Delta u_1^\varepsilon = 0 \text{ in } \Omega_1^\varepsilon,\; u_1^\varepsilon = u_2^\varepsilon \text{ on } \Gamma_\varepsilon. \quad (70)$$

Problem (69) is a Neumann problem. Since $\int_\Sigma f = 0$ and:

$$\int_\Sigma \partial_{n_1} q_1 u_1^\varepsilon = \int_{\Omega_1^\varepsilon} \Delta u_1^\varepsilon = 0, \quad (71)$$

the Neumann problem is compatible. As the solution of (69) is defined only up to a constant, we impose:

$$\int_\Sigma u_2^\varepsilon = 0, \quad (72)$$

to get uniqueness.

The following form of the topological derivative was obtained in the EIT problem [3]:

$$J(\Omega_2^\varepsilon) = J(\Omega) + \sum_{k=0}^{4} \int_\Sigma \mathcal{T}_{k,\varepsilon}(\hat{x}) + o\left(\varepsilon \left|S_\varepsilon^{N-1}\right|\right) + \mathcal{Z} + \mathcal{Y}$$

with

$$\mathcal{T}_{0,\varepsilon}(\hat{x}) = -2\varepsilon\left|S_\varepsilon^{N-1}\right| \alpha N^{-1} \nabla p(\hat{x}) \nabla u(\hat{x}),$$

$$\mathcal{T}_{1,\varepsilon}(\hat{x}) = \frac{\varepsilon^{2N}\alpha^2}{(N-1)^2} \sum_{i,j=1}^{N} \partial_i u_2(\hat{x}) \partial_j u_2(\hat{x}) I_{i,j}^{(1)},$$

$$\mathcal{T}_{2,\varepsilon}(\hat{x}) = -\frac{\varepsilon^{2(N+2)}\beta^2}{(N)^2} \sum_{i,j,k,l=1}^{N} \partial_{ij}^2 u_2(\hat{x}) \partial_{kl}^2 u_2(\hat{x}) I_{i,j,k,l}^{(2)},$$

$$\mathcal{T}_{3,\varepsilon}(\hat{x}) = -\frac{2\varepsilon^{2N+2}\alpha\beta}{(N-1)N} \sum_{i,j,k=1}^{N} \partial_k u_2(\hat{x}) \partial_{kij}^2 u_2(\hat{x}) I_{i,j,k}^{(12)},$$

$$\mathcal{T}_{4,\varepsilon}(\hat{x}) = -|\Sigma|^{-1}\left[\frac{\varepsilon^N\alpha}{N-1}\sum_{k=1}^{N}\partial_k u_2(\hat{x}) I_k^{(\lambda,1)} + \frac{\varepsilon^{N+2}\beta}{N}\sum_{i,j=1}^{N}\partial_{i,j}^2 u_2(\hat{x}) I_{i,j}^{(\lambda,2)}\right]$$

Some numerical results are presented below.

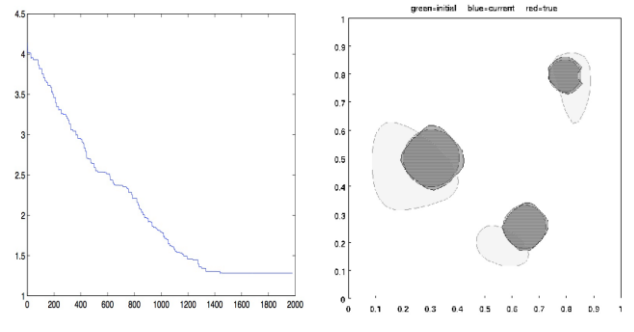

Fig. 11. First example after 2000 iterations, with 1% noise. Left: conductivity q_1 over the iterations. Right: initial domain Ω_1 after topological sensitivity (light gray); final domain Ω_1 upon termination of the algorithm (darker gray); true domain Ω_1 (stripes)

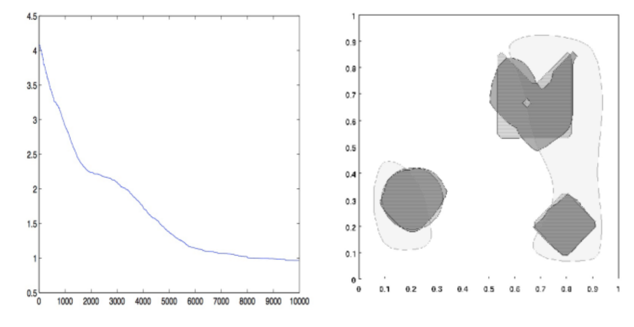

Fig. 12. Second example after 10000 iterations. Left: conductivity q_1 over the iterations. Right: initial domain Ω_1 after topological sensitivity (light gray); final domain Ω_1 upon termination of the algorithm (darker gray); true domain Ω_1 (stripes)

For more details we refer reader to [3], where the asymptotic analysis was provided for the EIT problem and the analytical for of the topological derivative was developed.

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